EXAM 2

Math 212, 2017 Summer Term 1, Clark Bray.

You have 75 minutes.

No notes, no books, no calculators.

YOU MUST SHOW ALL WORK AND EXPLAIN ALL REASONING
TO RECEIVE CREDIT. CLARITY WILL BE CONSIDERED IN GRADING.
All answers must be simplified. All of the policies and guidelines
on the class webpages are in effect on this exam.

Good luck!

Name Solutions

"I have adhered to the Duke Community Standard in completing this examination."

Signature: ____________

1. ________
2. ________
3. ________
4. ________
5. ________
6. ________

Total Score ________ (/100 points)

1. *(25 pts)* The function $f : \mathbb{R}^2 \to \mathbb{R}^1$, with $f(x,y) = 3e^{-(x^2+3y^2)}$, represents altitude (in miles) in terms of coordinates x and y (representing distances in miles, east and north (respectively) of a given fixed point) on a particular hill.

(a) Compute $\dfrac{\partial f}{\partial x}$ at $(2,1)$.

$$\frac{\partial f}{\partial x} = 3(-2x)e^{-(x^2+3y^2)} = -6xe^{-(x^2+3y^2)}$$

$$\frac{\partial f}{\partial x}(2,1) = -6(2)e^{-(2^2+3(1)^2)} = -12e^{-7}$$

(b) Find the unit vector indicating the "uphill" direction on the map at $(2,1)$.

$$\frac{\partial f}{\partial y} = -18ye^{-(x^2+3y^2)}$$

$$\frac{\partial f}{\partial y}(2,1) = -18e^{-7}$$

So $\nabla f(2,1) = \begin{pmatrix} -12e^{-7} \\ -18e^{-7} \end{pmatrix}$

The uphill direction is $\nabla f / \|\nabla f\|$,

$$\nabla f / \|\nabla f\| = \begin{pmatrix} -2 \\ -3 \end{pmatrix} \Big/ \sqrt{13}$$

(c) Bob is walking along a trail on this hill, and at the moment he comes to the point $(2,1)$ his velocity is $(3,2)$ (in units of miles per hour). How fast is he gaining altitude at that moment?

$$\frac{df}{dt} = D_{\vec{v}}f(2,1) = \nabla f \cdot \vec{v} = \begin{pmatrix} -12e^{-7} \\ -18e^{-7} \end{pmatrix} \cdot \begin{pmatrix} 3 \\ 2 \end{pmatrix} = -72e^{-7}$$

(d) How steep is the trail that Bob is walking on at $(2,1)$?

$$\text{Trail steepness} = D_{\vec{\mu}}f(2,1) \quad \left(\text{with } \vec{\mu} = \frac{\vec{v}}{\|\vec{v}\|} = \begin{pmatrix} 3 \\ 2 \end{pmatrix} \Big/ \sqrt{13}\right)$$

$$= \nabla f \cdot \vec{\mu}$$

$$= \begin{pmatrix} -12e^{-7} \\ -18e^{-7} \end{pmatrix} \cdot \begin{pmatrix} 3 \\ 2 \end{pmatrix} \Big/ \sqrt{13} = \frac{-72e^{-7}}{\sqrt{13}}$$

(e) How steep is the hillside itself (independent of the trail) at $(2,1)$?

$$\text{Hill steepness} = \|\nabla f\| = 6e^{-7}\|(-2,-3)\| = 6e^{-7}\sqrt{13}$$

2. *(15 pts)*

(a) Show that the function $g : \mathbb{R}^3 \to \mathbb{R}^2$ below is differentiable. *(Be sure to explain your reasoning fully.)*

$$g(x, y, z) = \begin{pmatrix} xye^z \\ \sin x - 3y - 4z \end{pmatrix} = \begin{pmatrix} g_1 \\ g_2 \end{pmatrix}$$

$$\frac{\partial g_1}{\partial x} = ye^z \qquad \frac{\partial g_1}{\partial y} = xe^z \qquad \frac{\partial g_1}{\partial z} = xye^z$$

$$\frac{\partial g_2}{\partial x} = \cos x \qquad \frac{\partial g_2}{\partial y} = -3 \qquad \frac{\partial g_2}{\partial z} = -4$$

All of these partials are continuous, so g is continuously differentiable, and thus it must also be differentiable.

(b) Find the matrix M that computes directional derivatives of g at the origin, by

$$D_{\vec{v}}g(\vec{0}) = M\vec{v}$$

M is the Jacobian matrix $J_{g,\vec{0}}$, whose elements are the partials above, evaluated at $\vec{0}$. So

$$M = J_{g,\vec{0}} = \begin{pmatrix} 0 & 0 & 0 \\ 1 & -3 & -4 \end{pmatrix}$$

3. *(20 pts)* Use the information below to compute $\dfrac{\partial p_1}{\partial y}(0,0)$.

$$f(x,y)=\begin{pmatrix}3x+y+1\\2x-3y-2\\3xy\end{pmatrix}\quad g(x,y,z)=\begin{pmatrix}2x-z\\4x+y+2z\end{pmatrix}\quad h(x,y)=\begin{pmatrix}y^2-x\\x+2y\end{pmatrix}$$

$$p(x,y)=(h\circ g\circ f)(x,y)=(p_1,p_2)$$

To avoid confusion of variables, we rewrite the functions g and h as:

$$g(u,v,w)=\begin{pmatrix}2u-w\\4u+v+2w\end{pmatrix}=\begin{pmatrix}s\\t\end{pmatrix},\quad h(s,t)=\begin{pmatrix}t^2-s\\s+2t\end{pmatrix}=\begin{pmatrix}p_1\\p_2\end{pmatrix}$$

making the composition

$$\begin{tikzcd}
\mathbb{R}^2 \arrow[r, "f"] \arrow[rrr, bend left, "p"] & \mathbb{R}^3 \arrow[r, "g"] & \mathbb{R}^2 \arrow[r, "h"] & \mathbb{R}^2 \\
\begin{pmatrix}x\\y\end{pmatrix} & \begin{pmatrix}u\\v\\w\end{pmatrix} & \begin{pmatrix}s\\t\end{pmatrix} & \begin{pmatrix}p_1\\p_2\end{pmatrix}
\end{tikzcd}$$

Then the chain rule gives us

$$J_p=J_hJ_gJ_f=\begin{pmatrix}-1&2t\\1&2\end{pmatrix}\begin{pmatrix}2&0&-1\\4&1&2\end{pmatrix}\begin{pmatrix}3&1\\2&-3\\3y&3x\end{pmatrix}$$

and

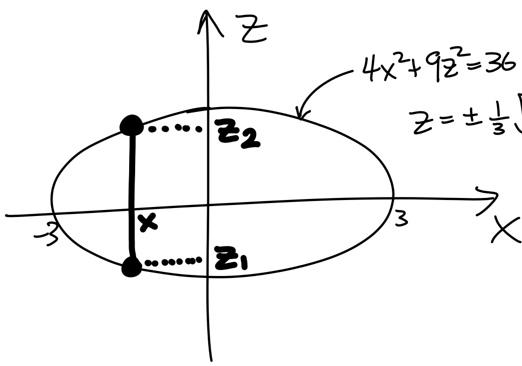
$$\begin{pmatrix}x\\y\end{pmatrix}=\begin{pmatrix}0\\0\end{pmatrix}\Rightarrow\begin{pmatrix}u\\v\\w\end{pmatrix}=\begin{pmatrix}1\\-2\\0\end{pmatrix}\Rightarrow\begin{pmatrix}s\\t\end{pmatrix}=\begin{pmatrix}2\\2\end{pmatrix}$$

so $$J_{p,(0,0)}=\begin{pmatrix}-1&4\\1&2\end{pmatrix}\begin{pmatrix}2&0&-1\\4&1&2\end{pmatrix}\begin{pmatrix}3&1\\2&-3\\0&0\end{pmatrix}=\begin{pmatrix}50&2\\34&4\end{pmatrix}$$

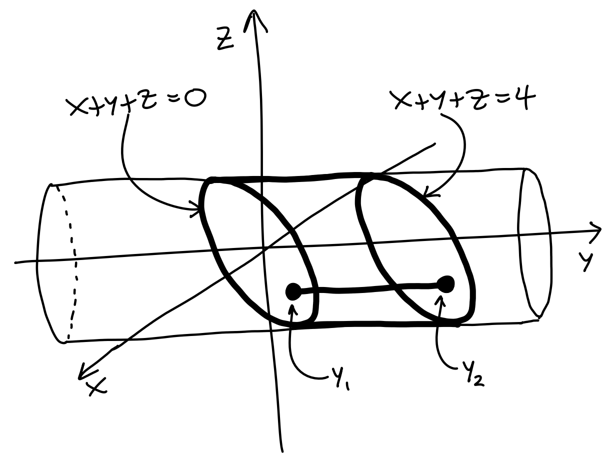
and $\dfrac{\partial p_1}{\partial y}(0,0)$ = entry in 1st row, 2nd col $=\boxed{2}$

4. *(20 pts)* Express as an iterated triple integral (but do not evaluate!) the mass in the region R bounded by the surfaces $4x^2 + 9z^2 = 36$, $x + y + z = 0$, and $x + y + z = 4$, with $\delta(x, y, z) = e^z$.

Projection to xz-plane:



R:



Choosing to slice as indicated by "dy dz dx", we can find the bounds on x and z from the projection, and on y from the solid. Then

$$m = \iiint \delta \, dV$$

$$= \int_{-3}^{3} \int_{-\frac{1}{3}\sqrt{36-4x^2}}^{\frac{1}{3}\sqrt{36-4x^2}} \int_{-x-z}^{4-x-z} e^z \, dy \, dz \, dx$$

Or, slicing by "dy dx dz",

$$= \int_{-2}^{2} \int_{-\frac{1}{2}\sqrt{36-9z^2}}^{\frac{1}{2}\sqrt{36-9z^2}} \int_{-x-z}^{4-x-z} e^z \, dy \, dx \, dz$$

5. *(5 pts)* Your friend Bob is interested in computing the volume of the half of the unit ball with $y \geq 0$. First he computes

$$\int_0^\pi \int_0^\pi \int_0^1 (1)\rho^2 \sin\phi \, d\rho \, d\phi \, d\theta$$

and gets $2\pi/3$. But then he decides to compute again using alternative spherical coordinates for the same domain, and computes

$$\int_\pi^{2\pi} \int_\pi^{2\pi} \int_0^1 (1)\rho^2 \sin\phi \, d\rho \, d\phi \, d\theta$$

and gets $-2\pi/3$. Can you help Bob resolve his apparent contradiction?

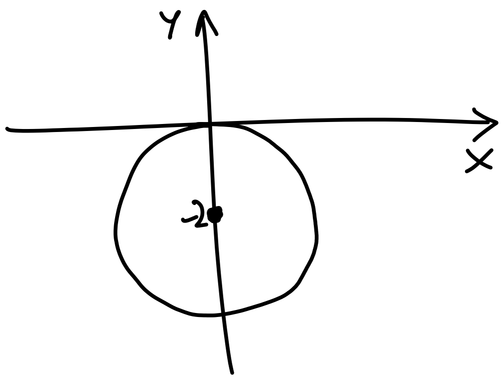
The stretching factor is $|\rho^2 \sin\phi|$. In his second calculation $\sin\phi$ is negative, so $|\rho^2\sin\phi| = -\rho^2 \sin\phi$, not $\rho^2 \sin\phi$.

So his integral should be

$$\int_\pi^{2\pi} \int_\pi^{2\pi} \int_0^1 (1)(-\rho^2 \sin\phi) \, d\rho \, d\phi \, d\theta$$

which is $\frac{2\pi}{3}$, as expected.

6. *(15 pts)* Express as an iterated integral in polar coordinates (but do not evaluate!) the integral $\iint_D x^2 - y^2 \, dx \, dy$, where D is the disk bounded by the circle of radius 2 with center at $(0, -2)$.



$$x^2 + (y+2)^2 = 2^2$$

$$x^2 + y^2 + 4y + 4 = 4$$

$$r^2 + 4r\sin\theta = 0$$

$$r = -4\sin\theta$$

$$\iint_D x^2 - y^2 \, dx\,dy = \int_\pi^{2\pi} \int_0^{-4\sin\theta} \left((r\cos\theta)^2 - (r\sin\theta)^2\right) r \, dr \, d\theta$$

$$= \int_\pi^{2\pi} \int_0^{-4\sin\theta} r^3 \cos 2\theta \, dr \, d\theta$$