EXAM 1

Math 212, 2018 Summer Term 2, Clark Bray.

Name: Solutions Section: ______ Student ID: ____________

GENERAL RULES

YOU MUST SHOW ALL WORK AND EXPLAIN ALL REASONING TO RECEIVE CREDIT. CLARITY WILL BE CONSIDERED IN GRADING.

No notes, no books, no calculators. Scratch paper is allowed, but (1) it must be from the instructor, (2) it must be returned with the exam, and (3) it will NOT be graded.

All answers must be reasonably simplified.

All of the policies and guidelines on the class webpages are in effect on this exam.

WRITING RULES

Do not write anything on the QR codes or near the staple.

Use black pen only – no pencils.

Work for a given question can be done ONLY on the front or back of the page the question is written on.

DUKE COMMUNITY STANDARD STATEMENT

"I have adhered to the Duke Community Standard in completing this examination."

Signature: ________________________

(Nothing on this page will be graded!)

1. *(20 pts)* The line $L \in \mathbb{R}^3$ is parametrized by $\vec{x}(t) = (3-t, 2t, t+1)$, and the line C has symmetric equations

$$\frac{x-1}{2} = \frac{y+1}{6} = \frac{z+3}{10}$$

(a) Do L and C intersect? If so, find the point of intersection; if not, explain how you know.

We would need a point generated by $\vec{x}(t)$ to satisfy the symmetric equations. So

$$\frac{(3-t)-1}{2} = \frac{2t+1}{6} = \frac{(t+1)+3}{10}$$

$$30-15t = 10t+5 = 3t+12$$

$$t=1 \qquad\qquad t=1$$

These equations are satisfied by $t=1$, so the lines do intersect and the point of intersection is $\vec{x}(1) = (2,2,2)$.

(b) Find the equation of the plane that contains L and is parallel to C.

$$\frac{x-1}{2} = \frac{y+1}{6} = \frac{z+3}{10} = s \implies \begin{aligned} x &= 1+2s \\ y &= -1+6s \\ z &= -3+10s \end{aligned}$$

So $(2,6,10)$, $\parallel$ to C, and $(-1,2,1)$, $\parallel$ to L, are both $\parallel$ to the plane, and we use as a normal vector

$$\vec{n} = \begin{pmatrix} 2 \\ 6 \\ 10 \end{pmatrix} \times \begin{pmatrix} -1 \\ 2 \\ 1 \end{pmatrix} = \begin{pmatrix} -14 \\ -12 \\ 10 \end{pmatrix}$$

Using the $\vec{x}_0 = (2,2,2)$, $\vec{n}\cdot\vec{x} = \vec{n}\cdot\vec{x}_0$ becomes

$$-14x - 12y + 10z = -32$$

$$\boxed{7x + 6y - 5z = 16}$$

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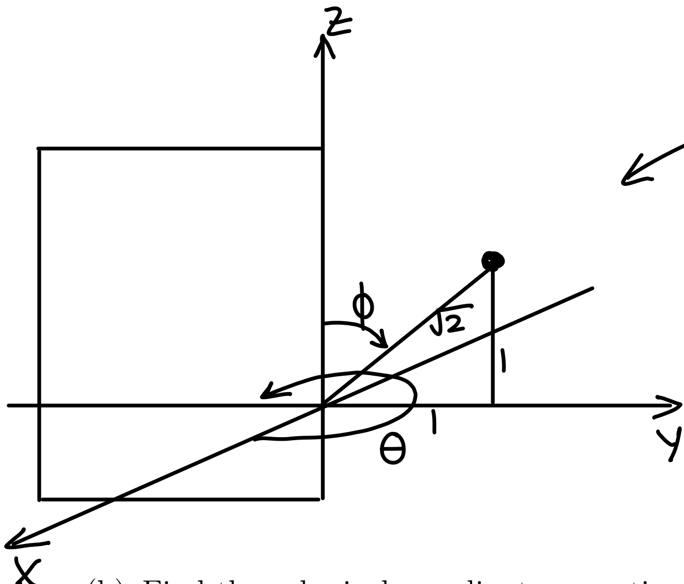
2. *(20 pts)*

(a) The point $(x, y, z) = (0, 1, 1)$ can be described in spherical coordinates in exactly one way with $\pi \le \theta \le 2\pi$ and $-\pi \le \phi \le 0$. Find these values of ρ, ϕ, and θ.



$\rho = \sqrt{2}$

$\phi = -\pi/4$

$\theta = \frac{3\pi}{2}$

(b) Find the spherical coordinates equation for the sphere of radius 1 centered at $(0, -1, 0)$.

$$x^2 + (y+1)^2 + z^2 = 1$$

$$x^2 + y^2 + z^2 + 2y + 1 = 1$$

$$\rho^2 + 2\rho \sin\phi \sin\theta = 0$$

$$\boxed{\rho = -2\sin\phi\sin\theta}$$

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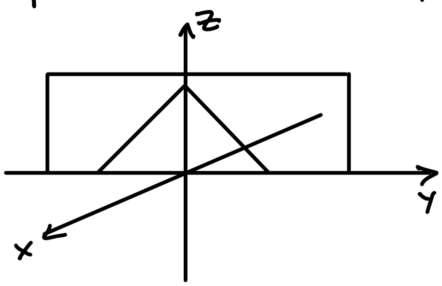
3. *(20 pts)*

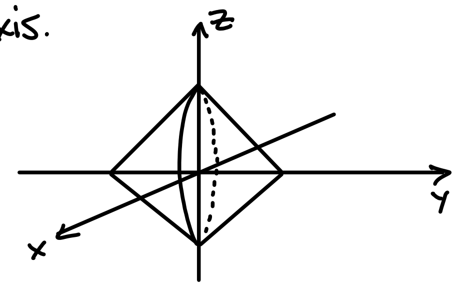
(a) The surface S has equation $\sqrt{x^2+z^2} = 1-|y|$. Give a clear description of simple geometric processes that would generate S.

Rot. symm. around y-axis ; equation in $\{x=0, z\geq 0\}$ half plane is $z = 1-|y|$. This is as shown below.



We get S by rotating this curve around the y-axis.

This is two cones attached at the base:



(b) Give an explicit, ordered series of geometric transformations that can be applied to the unit sphere to result in the surface with equation $4x^2 + (4y+8)^2 + z^2 = 1$.

$x^2+y^2+z^2=1$

$\downarrow$ "x" $\mapsto$ "$2x$" : ① Squish in x-dir. by factor of 2.

$4x^2+y^2+z^2=1$

$\downarrow$ "y" $\mapsto$ "$y+8$" : ② Translate in negative y-dir. by distance 8.

$4x^2+(y+8)^2+z^2=1$

$\downarrow$ "y" $\mapsto$ "$4y$" : ③ Squish in y-dir. by factor of 4.

$4x^2+(4y+8)^2+z^2=1$

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4. *(20 pts)* The surface S has equation $xe^y - ze^{xy} = ye^x$.

(a) Find a function $p : \mathbb{R}^a \to \mathbb{R}^b$ whose graph is S, and identify the values of a and b.

Equation is equivalent to

$$z = \frac{xe^y - ye^x}{e^{xy}}$$

This is the graph $z = p(x,y)$ of $p : \mathbb{R}^2 \to \mathbb{R}^1$ given by

$$p(x,y) = \frac{xe^y - ye^x}{e^{xy}}$$

(b) Find a function $q : \mathbb{R}^c \to \mathbb{R}^d$ for which a level set is S, and identify the values of c and d.

Equation is equivalent to

$$xe^y - ze^{xy} - ye^x = 0$$

This is the level set $q(x,y,z) = 0$ of $q : \mathbb{R}^3 \to \mathbb{R}^1$ given by

$$q(x,y,z) = xe^y - ze^{xy} - ye^x$$

(c) Find a function $r : \mathbb{R}^f \to \mathbb{R}^g$ that parametrizes S, and identify the values of f and g.

The graph parametrization is $r : \mathbb{R}^2 \to \mathbb{R}^3$ given by

$$r(u,v) = \left(u, v, \frac{ue^v - ve^u}{e^{uv}}\right)$$

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5. *(20 pts)* The function L is a linear transformation, and we are given

$$L\left(\begin{pmatrix}1\\0\end{pmatrix}\right) = \begin{pmatrix}3\\5\end{pmatrix} \quad \text{and} \quad L\left(\begin{pmatrix}1\\1\end{pmatrix}\right) = \begin{pmatrix}1\\9\end{pmatrix}$$

Find the matrix A that represents L. *(Hint: Start by writing* $(0,1)$ *as a linear combination of* $(1,0)$ *and* $(1,1)$*.)*

$\begin{pmatrix}0\\1\end{pmatrix} = \begin{pmatrix}1\\1\end{pmatrix} - \begin{pmatrix}1\\0\end{pmatrix}$, so by linearity we have

$$\begin{aligned} L\begin{pmatrix}0\\1\end{pmatrix} &= L\left(\begin{pmatrix}1\\1\end{pmatrix} - \begin{pmatrix}1\\0\end{pmatrix}\right) \\ &= L\begin{pmatrix}1\\1\end{pmatrix} - L\begin{pmatrix}1\\0\end{pmatrix} \\ &= \begin{pmatrix}1\\9\end{pmatrix} - \begin{pmatrix}3\\5\end{pmatrix} = \begin{pmatrix}-2\\4\end{pmatrix} \end{aligned}$$

The matrix representing L has columns $L\begin{pmatrix}1\\0\end{pmatrix}, L\begin{pmatrix}0\\1\end{pmatrix}$,

so

$$A = \begin{pmatrix}3 & -2\\5 & 4\end{pmatrix}$$

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