EXAM 2

Math 212, 2018 Summer Term 2, Clark Bray.

Name: Solutions Section:______ Student ID:____________

GENERAL RULES

YOU MUST SHOW ALL WORK AND EXPLAIN ALL REASONING TO RECEIVE CREDIT. CLARITY WILL BE CONSIDERED IN GRADING.

No notes, no books, no calculators. Scratch paper is allowed, but (1) it must be from the instructor, (2) it must be returned with the exam, and (3) it will NOT be graded.

All answers must be reasonably simplified.

All of the policies and guidelines on the class webpages are in effect on this exam.

WRITING RULES

Do not write anything on the QR codes or near the staple.

Use black pen only. You may use a pencil for initial sketches of diagrams, but the final sketch must be drawn over in black pen and you must wipe all erasure residue from the paper.

Work for a given question can be done ONLY on the front or back of the page the question is written on.

DUKE COMMUNITY STANDARD STATEMENT

"I have adhered to the Duke Community Standard in completing this examination."

Signature: ________________________

(Nothing on this page will be graded!)

1. *(20 pts)* The differentiable function $w : \mathbb{R}^2 \to \mathbb{R}^3$ has components $w(u,v) = \Big(w_1(u,v), w_2(u,v), w_3(u,v)\Big)$. We know that $w(1,2) = (3,4,5)$, and

$$J_{w,(1,2)} = \begin{pmatrix} 3 & 4 \\ 2 & 1 \\ 6 & 5 \end{pmatrix}$$

(a) Compute $\left.\frac{d}{dt}\right|_{t=0} w(3t+1, 2-5t)$.

$$= \left.\frac{d}{dt}\right|_{t=0} w\left(\begin{pmatrix}1\\2\end{pmatrix} + t\begin{pmatrix}3\\-5\end{pmatrix}\right) = D_{(3,-5)} w(1,2) = D_{w,(1,2)}\begin{pmatrix}3\\-5\end{pmatrix}$$

$$= J_{w,(1,2)}\begin{pmatrix}3\\-5\end{pmatrix} = \begin{pmatrix} 3 & 4 \\ 2 & 1 \\ 6 & 5 \end{pmatrix}\begin{pmatrix}3\\-5\end{pmatrix} = \begin{pmatrix}-11\\1\\-7\end{pmatrix}$$

(b) At $(1,2)$, in what direction in the uv plane is the function w_3 increasing the fastest, and what is the rate of change of w_3 with respect to distance in that direction?

$$J = \begin{pmatrix} 3 & 4 \\ 2 & 1 \\ 6 & 5 \end{pmatrix} = \begin{pmatrix} \partial w_1/\partial u & \partial w_1/\partial v \\ \partial w_2/\partial u & \partial w_2/\partial v \\ \partial w_3/\partial u & \partial w_3/\partial v \end{pmatrix}$$

bottom row $= \nabla w_3$

Dir. of fastest increase is the direction of ∇w_3, which is $\begin{pmatrix}6\\5\end{pmatrix}/\sqrt{61}$.

dw_3/ds in that direction is $\|\nabla w_3\| = \sqrt{61}$.

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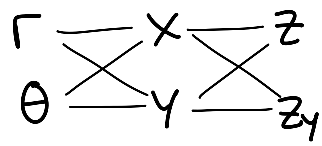
2. *(20 pts)* Suppose that z is a continuously twice differentiable function of x and y, which themselves are the usual functions of r and θ, and let $w = x\, z_y$. Compute $\dfrac{\partial w}{\partial \theta}$, writing your result in terms of only x, y, and the partials of z with respect to x and y.



$x = r\cos\theta$

$y = r\sin\theta$

$$\frac{\partial w}{\partial \theta} = \frac{\partial x}{\partial \theta} z_y + x \frac{\partial z_y}{\partial \theta}$$

$$= (-r\sin\theta) z_y + x\left(\frac{\partial z_y}{\partial x}\frac{\partial x}{\partial \theta} + \frac{\partial z_y}{\partial y}\frac{\partial y}{\partial \theta} \right)$$

$$= -z_y r\sin\theta + x\left(z_{xy}(-r\sin\theta) + z_{yy}(r\cos\theta) \right)$$

$$= -y z_y - xy z_{xy} + x^2 z_{yy}$$

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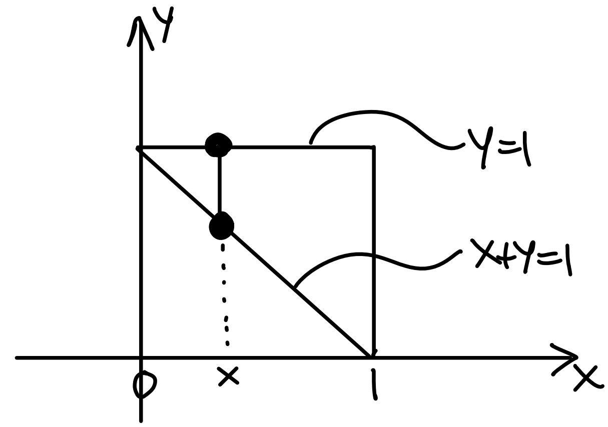
3. *(20 pts)* A pond is in the shape of a triangle in the xy-plane with vertices at $(1,1)$, $(0,1)$, and $(1,0)$. The depth in the pond depends on position by $h(x,y) = x$. Compute the total volume of water in the pond.



$$V = \iint dV$$

$$= \iint h \, dA$$

$$= \int_0^1 \int_{1-x}^1 x \, dy \, dx$$

$$= \int_0^1 \left(xy \right]_{y=1-x}^{y=1} dx$$

$$= \int_0^1 \left((x) - (x(1-x)) \right) dx$$

$$= \int_0^1 x^2 \, dx$$

$$= \frac{1}{3} x^3 \Big]_0^1 = \frac{1}{3}$$

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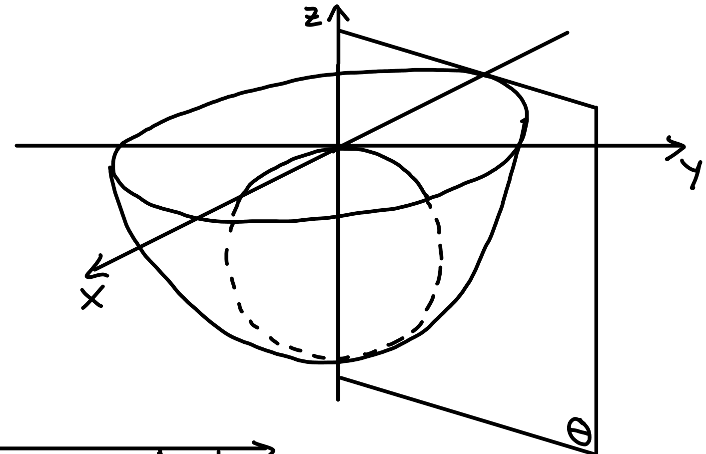
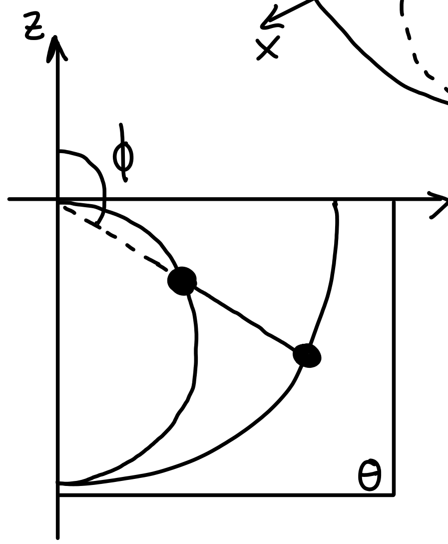
4. *(20 pts)* The region R is the collection of points in $\mathbb{R}^3$ that are below the xy-plane, inside the sphere of radius 4 centered at the origin, and outside the sphere of radius 2 centered at $(0, 0, -2)$.

Set up (but do not evaluate!) a triple iterated integral in spherical coordinates that represents $\iiint_R x\, dV$.



θ ranges from 0 to 2π.

θ cross section:



ϕ ranges from $\pi/2$ to π

inside sphere:

$$x^2+y^2+(z+2)^2=2^2$$

$$\rho^2+4z+4=4$$

$$\rho^2=-4\rho\cos\phi$$

$$\rho=-4\cos\phi$$

outside sphere:

$$\rho=4$$

$$\iiint x\, dV = \int_0^{2\pi}\int_{\pi/2}^{\pi}\int_{-4\cos\phi}^{4}(\rho\sin\phi\cos\theta)(\rho^2\sin\phi)\, d\rho\, d\phi\, d\theta$$

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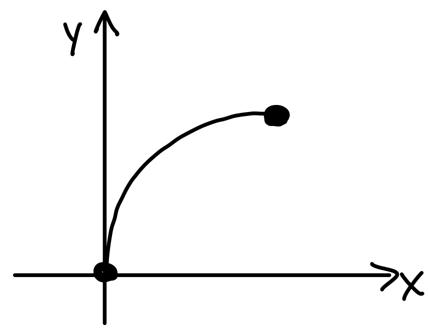
5. *(20 pts)* Bob's house is on a plot of land that we think of in the xy-plane, with distances measured in feet. His driveway is the upper-left quarter of the circle of radius 100 with center at $(100, 0)$ (it meets the road at the origin and his house at $(100, 100)$). The driveway is 12 feet wide.

A snowstorm leaves snow on the ground with depth (also in feet) given by $h(x, y) = 1 + y/100$, and the density of the snow is 11 pounds per cubic foot.

Bob plans to shovel his entire driveway. How many total pounds of snow does he have to move?

$$V = \int 12\, h\, ds$$

$$m = 11V = 132 \int h\, ds$$



$$\vec{x}(t) = \begin{pmatrix} 100 + 100\cos t \\ 100 \sin t \end{pmatrix}, \quad t \in \left[\frac{\pi}{2}, \pi\right]$$

$$\vec{x}'(t) = \begin{pmatrix} -100 \sin t \\ 100 \cos t \end{pmatrix}, \quad \|\vec{x}'\| = 100$$

$$m = 132 \int h\, ds$$

$$= 132 \int_{\pi/2}^{\pi} \left(1 + \frac{y}{100}\right) \|\vec{x}'\|\, dt$$

$$= 132 \int_{\pi/2}^{\pi} (1 + \sin t)(100)\, dt$$

$$= 13200 \int_{\pi/2}^{\pi} (1 + \sin t)\, dt$$

$$= 13200 \left(t - \cos t\right]_{\pi/2}^{\pi} = 13200\left(\frac{\pi}{2} + 1\right)$$

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