EXAM 1

Math 212, 2018-2019 Summer Term 1 (Marine Lab), Clark Bray.

You have 75 minutes.

No notes, no books, no calculators.

YOU MUST SHOW ALL WORK AND EXPLAIN ALL REASONING
TO RECEIVE CREDIT. CLARITY WILL BE CONSIDERED IN GRADING.
All answers must be simplified. All of the policies and guidelines
on the class webpages are in effect on this exam.

Good luck!

Name Solutions

1. ________

2. ________

3. ________

4. ________

5. ________

6. ________

7. ________

"I have adhered to the Duke Community
Standard in completing this
examination."

Signature: ________

Total Score ________ (/100 points)

1. *(15 pts)* The line L_1 has symmetric equations $2x-4=3y-3=z-3$, and the line L_2 is parametrized by $(x,y,z)=(1+t,4-3t,-1+4t)$.

$$= \underbrace{\begin{pmatrix}1\\4\\-1\end{pmatrix}}_{\vec{X}_2} + t\underbrace{\begin{pmatrix}1\\-3\\4\end{pmatrix}}_{\vec{V}_2}$$

(a) Do these lines intersect? If yes, find the point of intersection.

We need:

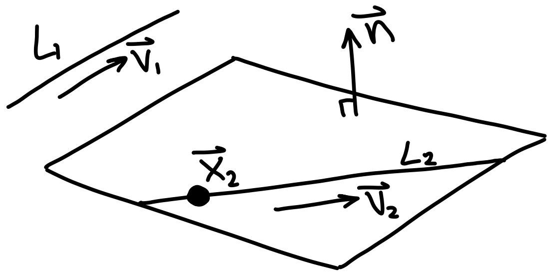
$$2(1+t)-4 = 3(4-3t)-3 = (-1+4t)-3$$

$$2t-2 = 9-9t = 4t-4$$

$$\Rightarrow t=1, \quad \begin{pmatrix}x\\y\\z\end{pmatrix}=\begin{pmatrix}2\\1\\3\end{pmatrix}$$

(b) Find the equation of the plane parallel to L_1 that contains L_2.

$$L_1: \quad 2x-4=3y-3=z-3=t \quad \Rightarrow \begin{pmatrix}x\\y\\z\end{pmatrix}=\begin{pmatrix}\frac{1}{2}t+2\\ \frac{1}{3}t+1\\ 1t+3\end{pmatrix}=\begin{pmatrix}2\\1\\3\end{pmatrix}+t\overbrace{\begin{pmatrix}\frac{1}{2}\\ \frac{1}{3}\\ 1\end{pmatrix}}^{\vec{V}_1}$$



Choose $n = 6\vec{V}_1 \times \vec{V}_2$

$$= \det\begin{pmatrix}\vec{e}_1 & \vec{e}_2 & \vec{e}_3\\ 3 & 2 & 6\\ 1 & -3 & 4\end{pmatrix} = \begin{pmatrix}26\\-6\\-11\end{pmatrix}$$

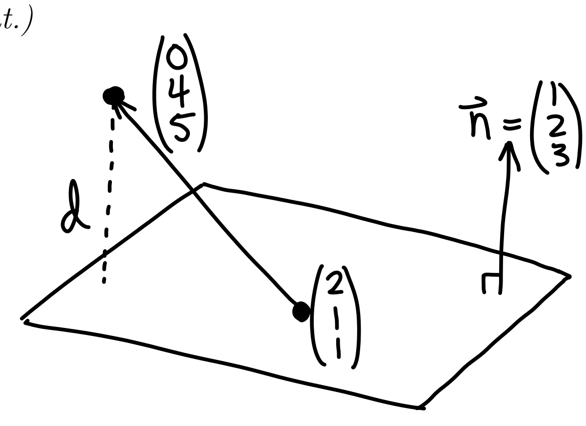
$$\vec{n}\cdot\vec{x}=\vec{n}\cdot\vec{X}_2=13 \quad \Rightarrow \quad 26x-6y-11z=13$$

2. *(5 pts)* The plane P has normal vector $(1,2,3)$ and contains the point $(2,1,1)$. Find the distance from P to $(0,4,5)$. *(Hint: This distance is a component.)*

$$d = \mathrm{comp}_{\vec{n}}\left(\begin{pmatrix}0\\4\\5\end{pmatrix}-\begin{pmatrix}2\\1\\1\end{pmatrix}\right)$$

$$= \begin{pmatrix}1\\2\\3\end{pmatrix}\cdot\begin{pmatrix}-2\\3\\4\end{pmatrix} \Big/ \|(1,2,3)\|$$

$$= 16/\sqrt{14}$$



3. *(10 pts)* Describe a geometric process by which you could generate the surface with equation $z = \ln(x^2 + y^2)$ from a standard curve of high school algebra.

This surface is rotationally symmetric around the z-axis, and the cross section in the half plane $\{y=0, x\geq 0\}$ has equation $z = \ln(x^2) = 2\ln x$, standard in high school algebra.

So the surface is generated by rotating this curve around the z-axis.

4. *(10 pts)* Find the equation of the surface that results from applying to the unit sphere the following sequence of transformations.

   (a) Stretch by a factor of 2 in the x direction;

   (b) Shift by a distance 4 in the positive x direction;

   (c) Shift by a distance 3 in the positive y direction;

   (d) Reflect through the xz-plane.

$$x^2+y^2+z^2=1 \xrightarrow[\text{"}x\text{"} \mapsto \text{"}\frac{x}{2}\text{"}]{(a)} \frac{x^2}{4}+y^2+z^2=1$$

$$\xrightarrow[\text{"}x\text{"} \mapsto \text{"}x-4\text{"}]{(b)} \frac{(x-4)^2}{4}+y^2+z^2=1$$

$$\xrightarrow[\text{"}y\text{"} \mapsto \text{"}y-3\text{"}]{(c)} \frac{(x-4)^2}{4}+(y-3)^2+z^2=1$$

$$\xrightarrow[\text{"}y\text{"} \mapsto \text{"}-y\text{"}]{(d)} \frac{(x-4)^2}{4}+(-y-3)^2+z^2=1$$

$$\frac{(x-4)^2}{4}+(y+3)^2+z^2=1$$

5. *(30 pts)* The surface S has equation $x^2z + z - x^3y = x^4y^3$.

(a) Is S a level set of any function f? If so, find such a function, its domain, and its target.

$$\iff x^2z + z - x^3y - x^4y^3 = 0$$

This is the $f=0$ level set of $f: \mathbb{R}^3 \to \mathbb{R}^1$ defined by

$$f(x,y) = x^2z + z - x^3y - x^4y^3$$

(b) Is S the graph of any function g? If so, find such a function, its domain, and its target.

$$x^2z + z - x^3y - x^4y^3 = 0$$

$$\iff (x^2+1)z - (x^3y + x^4y^3) = 0 \iff z = \frac{x^3y + x^4y^3}{x^2+1}$$

This is the graph $z = g(x,y)$ of $g: \mathbb{R}^2 \to \mathbb{R}^1$ defined by

$$g(x,y) = \frac{x^3y + x^4y^3}{x^2+1}$$

(c) Is S the image of any function h? If so, find such a function, its domain, and its target.

Using the graph parametrization, S is the image of $h: \mathbb{R}^2 \to \mathbb{R}^3$ defined by

$$h(u,v) = \begin{pmatrix} u \\ v \\ \dfrac{u^3v + u^4v^3}{u^2+1} \end{pmatrix}$$

6. *(15 pts)* Evaluate the following limit or show that it does not exist.

$$\lim_{(x,y)\to(0,0)} \frac{x^4}{(x^2+y^2)^2}$$

Along lines $y = mx$, parametrized by (t, mt), we have

$$\lim_{t\to 0} \frac{t^4}{(t^2+m^2t^2)^2}$$

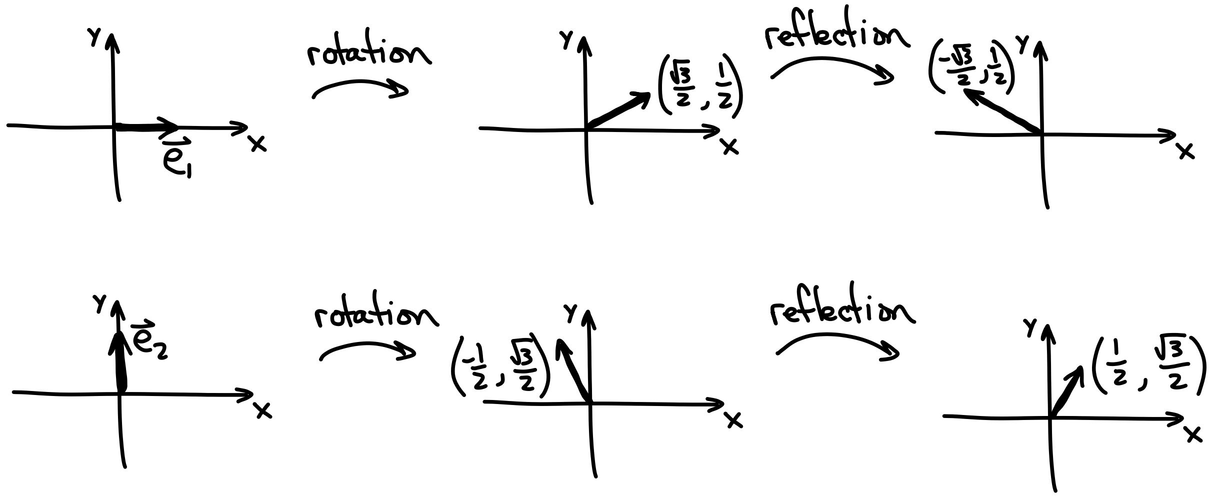
$$= \lim_{t\to 0} \frac{1}{(1+m^2)^2} = \frac{1}{(1+m^2)^2}$$

This gives different values along different lines.

So the original limit does not exist.

7. *(15 pts)* The linear transformation $L : \mathbb{R}^2 \to \mathbb{R}^2$ acts on vectors $\vec{v} \in \mathbb{R}^2$ by first rotating counter-clockwise around the origin by $\pi/6$, then reflecting over the y-axis.

(a) Find the matrix M that represents L.



$$M = \begin{pmatrix} L(\vec{e}_1) & L(\vec{e}_2) \end{pmatrix} = \begin{pmatrix} -\sqrt{3}/2 & 1/2 \\ 1/2 & \sqrt{3}/2 \end{pmatrix}$$

(b) Use an operation of matrix algebra to find the matrix that represents $L \circ L$.

$L \circ L$ has matrix $MM = M^2$.

$$\begin{pmatrix} -\sqrt{3}/2 & 1/2 \\ 1/2 & \sqrt{3}/2 \end{pmatrix}\begin{pmatrix} -\sqrt{3}/2 & 1/2 \\ 1/2 & \sqrt{3}/2 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$