EXAM 1

Math 212, 2019 Summer Term 2, Clark Bray.

Name: Solutions Section:______ Student ID:____________

GENERAL RULES

YOU MUST SHOW ALL WORK AND EXPLAIN ALL REASONING TO RECEIVE CREDIT. CLARITY WILL BE CONSIDERED IN GRADING.

No notes, no books, no calculators.

All answers must be reasonably simplified.

All of the policies and guidelines on the class webpages are in effect on this exam.

WRITING RULES

Do not write anything near the staple – this will be cut off.

Use black pen only. You may use a pencil for initial sketches of diagrams, but the final sketch must be drawn over in black pen and you must wipe all erasure residue from the paper.

Work for a given question can be done ONLY on the front or back of the page the question is written on. Room for scratch work is available on the back of this cover page, and on the two blank pages at the end of this packet; scratch work will NOT be graded.

DUKE COMMUNITY STANDARD STATEMENT

"I have adhered to the Duke Community Standard in completing this examination."

Signature: ________________________

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1. *(20 pts)*

(a) Given vectors $\vec{a} = (3,1,0)$, $\vec{b} = (5,4,3)$, $\vec{c} = (2,2,1)$, compute the volume of the parallelepiped $\|(\vec{a},\vec{b},\vec{c})$ and the area of the parallelogram $\|(\vec{a},\vec{b})$.

$$\det\begin{pmatrix} 3 & 1 & 0 \\ 5 & 4 & 3 \\ 2 & 2 & 1 \end{pmatrix} = 3(4\cdot 1 - 3\cdot 2) - 1(5\cdot 1 - 3\cdot 2) + 0 = -5$$

$$\text{volume} = |\det| = 5$$

$$\vec{a}\times\vec{b} = \det\begin{pmatrix} \vec{e}_1 & \vec{e}_2 & \vec{e}_3 \\ 3 & 1 & 0 \\ 5 & 4 & 3 \end{pmatrix} = \begin{pmatrix} 3 \\ -9 \\ 7 \end{pmatrix}$$

$$\text{area} = \|\vec{a}\times\vec{b}\| = \sqrt{3^2 + (-9)^2 + 7^2} = \sqrt{139}$$

(b) Use a determinant and related algebra to show that no list $\vec{v}\times\vec{w}, \vec{v}, \vec{w}$ of vectors in $\mathbb{R}^3$ can be in left hand order.

$$\det\begin{pmatrix} \vec{v}\times\vec{w} \\ \vec{v} \\ \vec{w} \end{pmatrix} = (\vec{v}\times\vec{w})\cdot(\vec{v}\times\vec{w}) = \|\vec{v}\times\vec{w}\|^2 \geq 0$$

Being never negative, the list cannot be in left hand order.

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2. *(30 pts)* The plane P in $\mathbb{R}^3$ is parallel to $3x - y + 2z = 0$ and passes through the point $(1, 1, 1)$.

(a) Find a function f whose graph is P, and identify its domain and target.

$\vec{n} = (3,-1,2)$, $\vec{X}_0 = (1,1,1)$ $\Longrightarrow$ $\vec{n} \cdot \vec{X}_0 = 4$

Equation of P is $3x - y + 2z = 4$

Solving for z: $z = \dfrac{4 - 3x + y}{2}$

So P is the graph $z = f(x,y)$ of $f : \mathbb{R}^2 \to \mathbb{R}^1$ defined by $f(x,y) = \dfrac{4 - 3x + y}{2}$

(b) Find a function g one of whose level sets is P, and identify its domain and target.

P is the level set $g = 4$ of $g : \mathbb{R}^3 \to \mathbb{R}^1$ defined by

$$g(x,y,z) = 3x - y + 2z$$

(c) Find a function h that parametrizes P, and identify its domain and target.

Using the graph parametrization from f, P is parametrized by $h : \mathbb{R}^2 \to \mathbb{R}^3$,

$$h(u,v) = \begin{pmatrix} u \\ v \\ \dfrac{4 - 3u + v}{2} \end{pmatrix}$$

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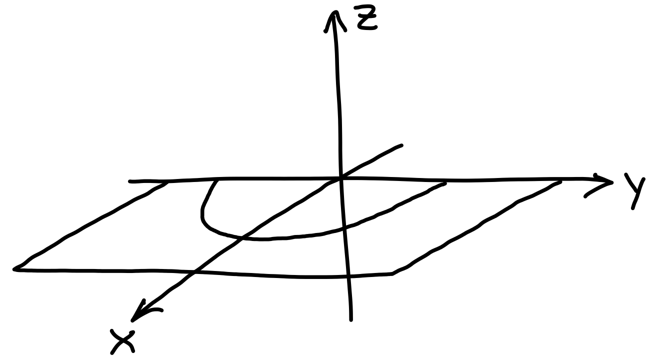
3. *(20 pts)* Bob is considering the surface S with equation $\sqrt{x^2 + z^2} = 4 - y^2$. He feels that this is the rotation around the y-axis of a parabola, which is unbounded, so S should also be unbounded; but he also feels that the right side of the equation should have to be positive, which suggests a bound on y and that therefore S should also be bounded. He is confused by this seeming contradiction.

(a) Help Bob resolve this seeming contradiction.

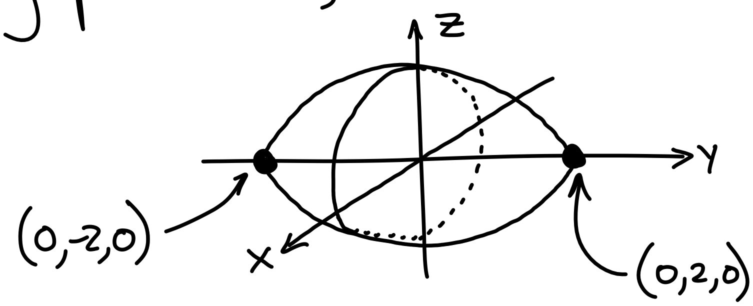
S is a rotation around the y-axis of the portion of the parabola $x = 4 - y^2$ in the half plane $z = 0$, $x \geq 0$.

This portion is bounded.



(b) What does S look like? (Explain in any combination of pictures and words, as best you can.)

Rotating per above, we have a "football":



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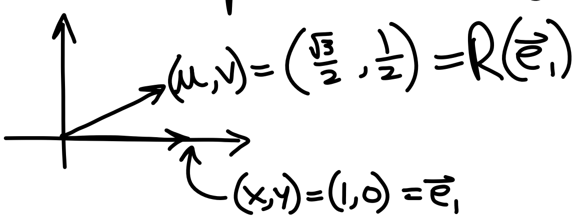
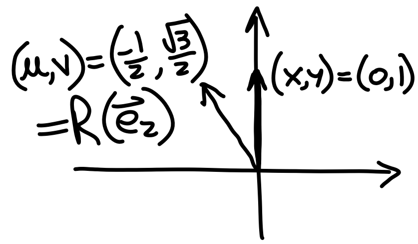
4. *(20 pts)*

(a) Find an equation for the linear transformation $R(x, y) = (u, v)$ that makes (u, v) the result of rotating (x, y) counterclockwise by the angle $\pi/6$.

We compute the images of the standard basis vectors:



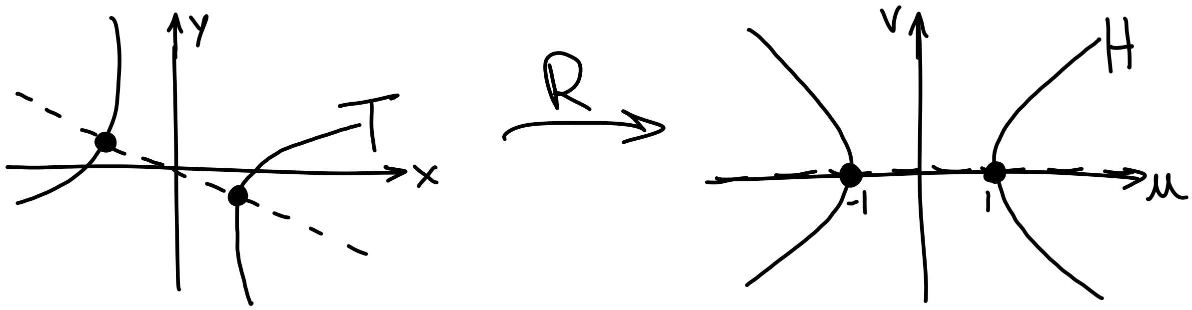


$$R(x,y) = x R(\vec{e}_1) + y R(\vec{e}_2)$$

$$= x \begin{pmatrix} \sqrt{3}/2 \\ 1/2 \end{pmatrix} + y \begin{pmatrix} -1/2 \\ \sqrt{3}/2 \end{pmatrix}$$

$$= \begin{pmatrix} \frac{\sqrt{3}}{2}x - \frac{1}{2}y \\ \frac{1}{2}x + \frac{\sqrt{3}}{2}y \end{pmatrix} = \begin{pmatrix} u \\ v \end{pmatrix}$$

(b) The equation $u^2 - v^2 = 1$ is a hyperbola H in the uv-plane that has vertices at $(u, v) = (\pm 1, 0)$. Use this fact and the equations in the linear transformation from part (a) to find the equation of a "tilted" hyperbola T in the xy-plane.



$$\left(\frac{\sqrt{3}}{2}x - \frac{1}{2}y\right)^2 - \left(\frac{1}{2}x + \frac{\sqrt{3}}{2}y\right)^2 = 1 \Rightarrow \frac{1}{2}x^2 - \frac{1}{2}y^2 - \sqrt{3}\,xy = 1$$

(c) Find the vertices of T, and explain your reasoning.

Points on T are rotated cc wise to points on H.

So H is rotated cwise to make T.

Vertices are $\pm\left(\frac{\sqrt{3}}{2}, \frac{-1}{2}\right)$.

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5. *(10 pts)* Compute the second column of the product below by using the interpretation as linear combinations of columns. (Do NOT use dot products!)

$$M = \begin{pmatrix} 1 & 4 & 0 \\ 3 & 0 & 3 \\ 2 & 1 & 5 \end{pmatrix} \begin{pmatrix} 2 & 1 & 2 \\ 5 & 3 & 1 \\ 1 & 4 & 1 \end{pmatrix}$$

The second column is the linear combination of the columns of the left matrix, using the second column of the right matrix as coefficients.

$$\vec{m}_2 = 1 \begin{pmatrix} 1 \\ 3 \\ 2 \end{pmatrix} + 3 \begin{pmatrix} 4 \\ 0 \\ 1 \end{pmatrix} + 4 \begin{pmatrix} 0 \\ 3 \\ 5 \end{pmatrix} = \begin{pmatrix} 13 \\ 15 \\ 25 \end{pmatrix}$$

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