EXAM 2

Math 212, 2019 Summer Term 2, Clark Bray.

Name: Solutions Section:______ Student ID:____________

GENERAL RULES

YOU MUST SHOW ALL WORK AND EXPLAIN ALL REASONING TO RECEIVE CREDIT. CLARITY WILL BE CONSIDERED IN GRADING.

No notes, no books, no calculators.

All answers must be reasonably simplified.

All of the policies and guidelines on the class webpages are in effect on this exam.

WRITING RULES

Do not write anything near the staple – this will be cut off.

Use black pen only. You may use a pencil for initial sketches of diagrams, but the final sketch must be drawn over in black pen and you must wipe all erasure residue from the paper.

Work for a given question can be done ONLY on the front or back of the page the question is written on. Room for scratch work is available on the back of this cover page, and on the two blank pages at the end of this packet; scratch work will NOT be graded.

DUKE COMMUNITY STANDARD STATEMENT

"I have adhered to the Duke Community Standard in completing this examination."

Signature: ________________________

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1. *(20 pts)* Suppose that Bob is in $\mathbb{R}^2$ at the point $\vec{a}$, and is interested in the values of the differentiable function $f : \mathbb{R}^2 \to \mathbb{R}^1$.

   (a) At first, Bob is planning to move from $\vec{a}$ with velocity $\vec{v}$. He then learns that $D_{\vec{v}}f(\vec{a}) = 6$, $\|\nabla f\| = 3$, and $\|\vec{v}\| = 4$. By what angle would Bob need to change the direction of his planned velocity (keeping speed the same) in order to maximize the rate of change of f?

   Direction of ∇f = direction of fastest increase

   So we want the angle between $\vec{v}$ and ∇f.

   $$6 = D_{\vec{v}}f(\vec{a}) = \nabla f \cdot \vec{v} = \|\nabla f\| \, \|\vec{v}\| \cos\theta$$
   $$= 3 \cdot 4 \cos\theta$$

   $$\Rightarrow \cos\theta = \frac{1}{2} \Rightarrow \theta = \pi/3.$$

   (b) Assuming that Bob makes the adjustment described in (a) above, what would be the rate of change of f with respect to time?

   New velocity $\vec{w}$ is $\parallel$ to ∇f, and $\|\vec{w}\| = 4$.

   $$\frac{df}{dt} = D_{\vec{w}}f(\vec{a}) = \nabla f \cdot \vec{w} = \|\nabla f\| \, \|\vec{w}\| \cos(0)$$
   $$= 3 \cdot 4 \cdot 1$$
   $$= 12.$$

   (c) Instead assuming that Bob keeps his originally planned velocity $\vec{v}$, what would be the rate of change of f with respect to distance traveled?

   $$\frac{df}{ds} = D_{\vec{u}}f(\vec{a}), \text{ where } \vec{u} = \frac{\vec{v}}{\|\vec{v}\|}.$$

   $$= \frac{1}{\|\vec{v}\|} D_{\vec{v}}f(\vec{a})$$

   $$= \frac{1}{4}(6) = \frac{3}{2}.$$

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2. *(20 pts)*

(a) What is the definition of the directional derivative of the function b at the point $\vec{p}$ with velocity $\vec{d}$?

$$D_{\vec{d}}\, b(\vec{p}) = \left.\frac{d}{dt}\right|_{t=0} b(\vec{p} + t\vec{d})$$

(b) Show that the function $f : \mathbb{R}^2 \to \mathbb{R}^2$ defined by $f(x, y) = (x^2 y e^{xy}, xye^{x^3}) = (f_1, f_2)$ is differentiable at the point $\vec{a} = (0, 1)$, and find the matrix representing the linear transformation T for which

$$D_{\vec{v}} f(\vec{a}) = T(\vec{v})$$

$$\frac{\partial f_1}{\partial x} = 2xye^{xy} + x^2y^2e^{xy} \qquad \frac{\partial f_1}{\partial y} = x^2e^{xy} + x^3y\,e^{xy}$$

$$\frac{\partial f_2}{\partial x} = y\,e^{x^3} + 3x^3y\,e^{x^3} \qquad \frac{\partial f_2}{\partial y} = xe^{x^3}$$

These partials are all sums, products, and compositions of known continuous functions, so they are continuous. So f is continuously differentiable and thus differentiable.

$T = D_{f,\vec{a}}$, so the matrix in question is

$$J_{f,\vec{a}} = \left.\begin{pmatrix} \frac{\partial f_1}{\partial x} & \frac{\partial f_1}{\partial y} \\ \frac{\partial f_2}{\partial x} & \frac{\partial f_2}{\partial y} \end{pmatrix}\right|_{\vec{a}}$$

$$= \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$$

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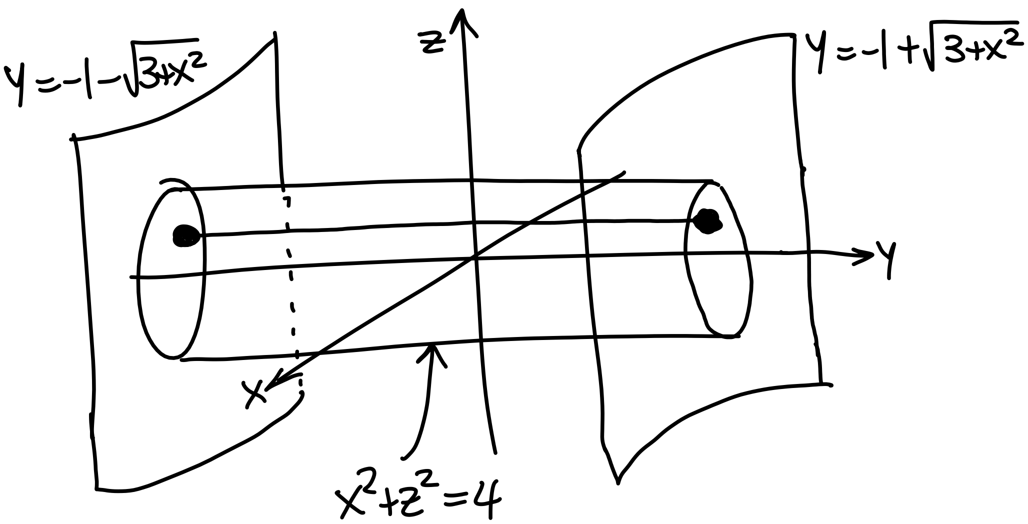
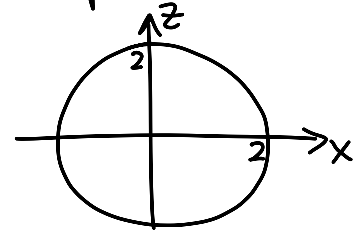
3. *(20 pts)* The solid T is the set of all points in $\mathbb{R}^3$ with $x^2 + z^2 \le 4$ and $(y+1)^2 \le 3 + x^2$. Set up, but do not evaluate, an iterated integral to compute $\iiint_T y^2\, dV$. *(Suggestion: note that "$a^2 \le b^2$" is equivalent to "$-|b| \le a \le |b|$".)*

$$(y+1)^2 \le 3+x^2 \iff -\sqrt{3+x^2} \le y+1 \le \sqrt{3+x^2}$$

$$\iff -1-\sqrt{3+x^2} \le y \le -1+\sqrt{3+x^2}$$



Projection to xz-plane:



$$\iiint y^2\, dV = \int_{-2}^{2} \int_{-\sqrt{4-z^2}}^{\sqrt{4-z^2}} \int_{-1-\sqrt{3+x^2}}^{-1+\sqrt{3+x^2}} y^2\, dy\, dx\, dz$$

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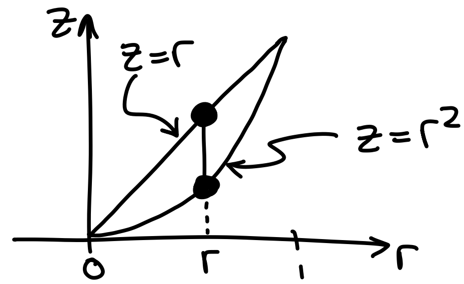
4. *(20 pts)* The region $D \subset \mathbb{R}^3$ is bounded between the graphs of the functions $f(x,y) = (x^2+y^2)^{1/2}$ and $g(x,y) = x^2+y^2$, and mass is distributed through D with density $\delta(x,y,z) = (x^2+y^2+z^2)$. Compute the total mass in D.

In cylindrical coordinates with $r \geq 0$, the surfaces are $z=r$ and $z=r^2$. Rotationally symmetric around the z-axis, the cross sections in a θ-slice are



$$m = \iiint_D \delta \, dV = \int_0^{2\pi}\int_0^1\int_{r^2}^{r} (r^2+z^2)\, r \, dz\, dr\, d\theta$$

$$= \int_0^{2\pi}\int_0^1\int_{r^2}^{r} r^3 + z^2 r \, dz\, dr\, d\theta$$

$$= \int_0^{2\pi}\int_0^1 r^3 z + \frac{1}{3}z^3 r \Big]_{z=r^2}^{z=r} dr\, d\theta$$

$$= \int_0^{2\pi}\int_0^1 \left(\frac{4}{3}r^4\right) - \left(r^5 + \frac{1}{3}r^7\right) dr\, d\theta$$

$$= \int_0^{2\pi}\left(\frac{4}{15}r^5 - \frac{1}{6}r^6 - \frac{1}{24}r^8\Big]_{r=0}^{r=1} d\theta\right.$$

$$= \int_0^{2\pi}\left(\frac{4}{15} - \frac{1}{6} - \frac{1}{24} - 0\right) d\theta$$

$$= \int_0^{2\pi} \frac{32-20-5}{120}\, d\theta = \frac{7\pi}{60}$$

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5. *(20 pts)* A valley is described by the equation $z = x^2$, sitting over the domain $[-1, 3] \times [2, 5]$ in the xy-plane. A certain type of weed is growing on the sides of the valley, with the number of weeds per unit area given by $(10 - z)^2$. Set up, but do not evaluate, an iterated integral that represents the total number of weeds growing in this valley.

Graph parametrization is

$$\vec{X}(u,v) = \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} u \\ v \\ u^2 \end{pmatrix} \qquad \vec{X}_u = \begin{pmatrix} 1 \\ 0 \\ 2u \end{pmatrix}, \ \vec{X}_v = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$$

$$\vec{N} = \vec{X}_u \times \vec{X}_v = \begin{pmatrix} -2u \\ 0 \\ 1 \end{pmatrix} \implies \|\vec{N}\| = \sqrt{1+4u^2}$$

Then

$$\# \text{ of weeds} = \iint_S (10-z)^2 \, dS$$

$$= \iint (10-z)^2 \, \|\vec{N}\| \, du \, dv$$

$$= \int_2^5 \int_{-1}^3 (10-u^2)^2 \sqrt{1+4u^2} \, du \, dv$$

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