EXAM 2

Math 212, 2020 Summer Term 1, Clark Bray.

Name: Solutions Section:______ Student ID:____________

GENERAL RULES

YOU MUST SHOW ALL WORK AND EXPLAIN ALL REASONING TO RECEIVE CREDIT. CLARITY WILL BE CONSIDERED IN GRADING.

No calculators.

All answers must be reasonably simplified.

All of the policies and guidelines on the class webpages are in effect on this exam.

WRITING RULES

Use black pen only. You may use a pencil for initial sketches of diagrams, but the final sketch must be drawn over in black pen and you must wipe all erasure residue from the paper.

DUKE COMMUNITY STANDARD STATEMENT

"I have adhered to the Duke Community Standard in completing this examination."

Signature: ________________________

(Scratch space. Nothing on this page will be graded!)

1. *(10 pts)* Consider the function $f : \mathbb{R}^2 \to \mathbb{R}^2$ defined by $f(x, y) = (x + |y|, y)$. Is f directional linear at the origin? Prove or disprove.

On $x=0$, $f(0,y) = (|y|, y)$. If it were to exist we would need

$$\frac{\partial}{\partial y}\Big|_{(0,0)} f(x,y) = \frac{d}{dt}\Big|_{t=0} (|t|, t)$$

$$= \left(\frac{d}{dt}\Big|_{t=0} |t| , \frac{d}{dt}\Big|_{t=0} t \right)$$

But $\frac{d}{dt}\Big|_{t=0} |t|$ does not exist. So at $(0,0)$ f is not partial differentiable, and thus not directional linear.

2. *(10 pts)* The function $g : \mathbb{R}^2 \to \mathbb{R}^1$ is differentiable, and for the vectors $\vec{v} = (1, 2)$ and $\vec{w} = (0, 2)$ we know the directional derivatives $D_{\vec{v}}g(\vec{a}) = 5$ and $D_{\vec{w}}g(\vec{a}) = 3$. Find the Jacobian matrix $J_{g,\vec{a}}$.

$\vec{e}_1 = \vec{v} - \vec{w}$, $\vec{e}_2 = \frac{\vec{w}}{2}$

$$\frac{\partial g}{\partial x} = J_g \vec{e}_1 = J_g(\vec{v} - \vec{w}) = D_{\vec{v}}g - D_{\vec{w}}g = 2$$

$$\frac{\partial g}{\partial y} = J_g \vec{e}_2 = J_g\left(\frac{\vec{w}}{2}\right) = \frac{1}{2} D_{\vec{w}}g = \frac{3}{2}$$

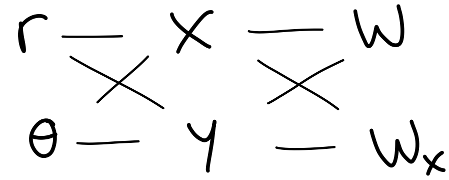
Then

$$J_{g,\vec{a}} = \begin{pmatrix} \frac{\partial g}{\partial x}(\vec{a}) & \frac{\partial g}{\partial y}(\vec{a}) \end{pmatrix} = \begin{pmatrix} 2 & \frac{3}{2} \end{pmatrix}$$

(extra space for questions from other side)

3. *(10 pts)* Suppose that x and y are the usual functions of the polar coordinates r and θ, and w is continuously twice differentiable function of x and y. Find a fully simplified expression for

$$\frac{\partial}{\partial \theta}(w_x w)$$



$$\frac{\partial}{\partial \theta}(w_x w) = \left(\frac{\partial w_x}{\partial \theta}\right)(w) + (w_x)\left(\frac{\partial w}{\partial \theta}\right)$$

$$= \left(\frac{\partial w_x}{\partial x}\frac{\partial x}{\partial \theta} + \frac{\partial w_x}{\partial y}\frac{\partial y}{\partial \theta}\right)(w) + (w_x)\left(\frac{\partial w}{\partial x}\frac{\partial x}{\partial \theta} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial \theta}\right)$$

$$= \left(w_{xx}(-r\sin\theta) + w_{xy}(r\cos\theta)\right)w + w_x\left(w_x(-r\sin\theta) + w_y(r\cos\theta)\right)$$

$$= (r\cos\theta)\left(w_{xy}w + w_x w_y\right) - (r\sin\theta)\left(w_{xx}w + w_x w_x\right)$$

4. *(10 pts)* The function $f : \mathbb{R}^3 \to \mathbb{R}^1$ is continuously differentiable, and we know that at the point $\vec{a}$ we have $\nabla f(\vec{a}) = (2, 1, 0)$.

On the level set passing through $\vec{a}$, decide which of $\frac{\partial x}{\partial y}$ and $\frac{\partial z}{\partial x}$ must exist and compute if possible.

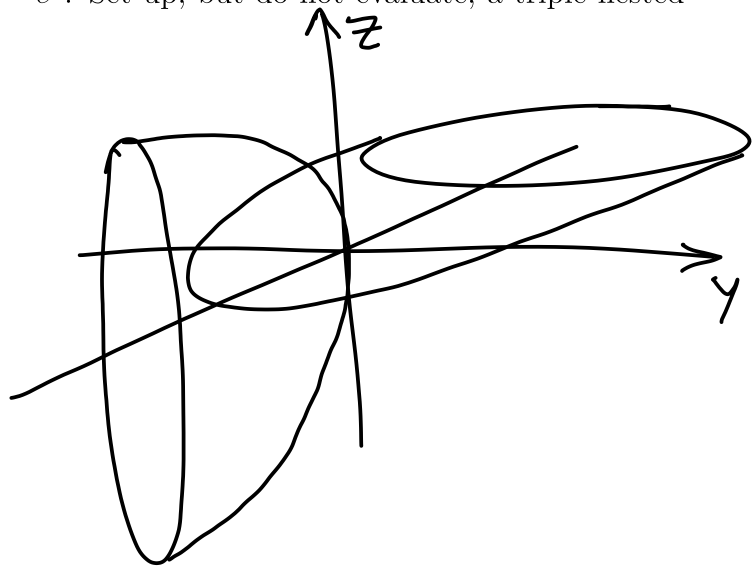
$\frac{\partial f}{\partial x}(\vec{a}) = 2 \neq 0$, so x is a differentiable function of y and z locally, and

$$\frac{\partial x}{\partial y} = -\frac{\partial f/\partial y}{\partial f/\partial x} = \frac{-1}{2}$$

$\frac{\partial f}{\partial z} = 0$, so z need not be a function of x and y locally.

(extra space for questions from other side)

5. *(15 pts)* The domain D is bounded by the surfaces $x - 2y^2 - z^2 = 0$ and $x + y^2 + 2z^2 = 12$, and mass is distributed through D with density $\delta = e^x$. Set up, but do not evaluate, a triple nested integral representing $\iiint_D e^x \, dV$.



Proj. to the yz-plane where

$x_1 = 2y^2 + z^2$

$x_2 = 12 - y^2 - 2z^2$

are equal. So

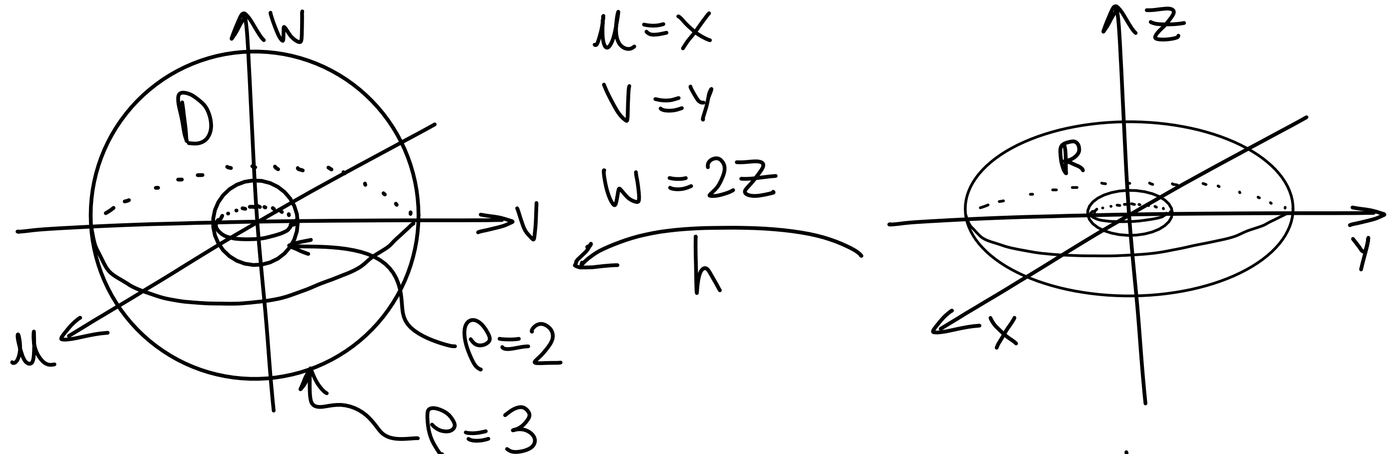
$$3y^2 + 3z^2 = 12 \implies y^2 + z^2 = 4$$

Slicing with dydz on the outside, we have

$$\iiint_D e^x \, dV = \int_{-2}^{2} \int_{-\sqrt{4-z^2}}^{\sqrt{4-z^2}} \int_{2y^2+z^2}^{12-y^2-2z^2} e^x \, dx \, dy \, dz$$

(extra space for questions from other side)

6. *(25 pts)* The ellipsoids in xyz-space have equations $x^2 + y^2 + 4z^2 = 4$ and $x^2 + y^2 + 4z^2 = 9$. The solid region between these two ellipsoids is R. Compute the integral over R of the function $g(x, y, z) = x^2 + y^2 + 4z^2 - x - 1$. *(Hint: Use a change of variables to turn the domain into a more convenient shape.)*



$$\frac{\partial(u,v,w)}{\partial(x,y,z)} = \det\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{pmatrix} = 2 \Rightarrow \left|\frac{\partial(x,y,z)}{\partial(u,v,w)}\right| = \frac{1}{2}$$

$$\iiint_R g(x,y,z)\,dx\,dy\,dz = \iint_D (x^2+y^2+4z^2-x-1)\left|\frac{\partial(x,y,z)}{\partial(u,v,w)}\right| du\,dv\,dw$$

$$= \iiint_D (u^2+v^2+w^2-u-1)\left(\frac{1}{2}\right) du\,dv\,dw$$

$$= \underbrace{\iiint_D \frac{1}{2}\rho^2\,\rho^2\sin\phi\,d\rho\,d\phi\,d\theta}_{A}\ \underbrace{-\frac{1}{2}\iiint_D u\,dV}_{B}\ \underbrace{-\frac{1}{2}\iiint_D 1\,dV}_{C}$$

$$A = \int_0^{2\pi}\int_0^{\pi}\int_2^3 \frac{1}{2}\rho^4\sin\phi\,d\rho\,d\phi\,d\theta = \int_0^{2\pi}\int_0^{\pi} \frac{1}{10}\rho^5\sin\phi\Big]_{\rho=2}^{\rho=3} d\phi\,d\theta$$

$$= \int_0^{2\pi}\int_0^{\pi} \frac{211}{10}\sin\phi\,d\phi\,d\theta = \frac{844\pi}{10} = \frac{422\pi}{5}$$

(extra space for questions from other side)

$B = -\frac{1}{2}\iiint_D u \, dV$, consider reflection through the vw-plane, $R(u,v,w) = (-u,v,w)$.

$f(u,v,w) = u$ has odd symmetry because

$f(R(u,v,w)) = f(-u,v,w) = -u = -f(u,v,w)$

And D is symmetric through the same plane.

So $B = 0$.

$$C = -\frac{1}{2}\iiint_D 1 \, dV = -\frac{1}{2}(\text{volume of } D)$$

$$= -\frac{1}{2}\left(\frac{4}{3}\pi(3)^3 - \frac{4}{3}\pi(2)^3\right)$$

$$= -\frac{38\pi}{3}$$

So the integral is

$$A + B + C = \frac{422\pi}{5} - \frac{38\pi}{3}$$

7. *(20 pts)* The plane P has equation $3x - 2y + 5z = 1$, and the surface S is the part of P with $0 \le y \le 1$ and $0 \le z \le 2$. Compute the integral over S of the function $f(x, y, z) = y$.

We parametrize using $x = \frac{1+2y-5z}{3}$ to get

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} (1+2u-5v)/3 \\ u \\ v \end{pmatrix} \Rightarrow X_u = \begin{pmatrix} 2/3 \\ 1 \\ 0 \end{pmatrix}, \quad \vec{X}_v = \begin{pmatrix} -5/3 \\ 0 \\ 1 \end{pmatrix}$$

$$\Rightarrow \vec{N} = \begin{pmatrix} 1 \\ -2/3 \\ 5/3 \end{pmatrix} \Rightarrow \|\vec{N}\| = \frac{\sqrt{38}}{3}$$

Then $$\iint_S y \, dS = \int_0^2 \int_0^1 y \, \|\vec{N}\| \, du \, dv$$

$$= \int_0^2 \int_0^1 \frac{\sqrt{38}}{3} u \, du \, dv$$

$$= \frac{\sqrt{38}}{6} \int_0^2 \left(u^2 \right]_{u=0}^{u=1} dv$$

$$= \frac{\sqrt{38}}{3}$$

(extra space for questions from other side)

(Scratch space. Nothing on this page will be graded!)

(Scratch space. Nothing on this page will be graded!)

(Scratch space. Nothing on this page will be graded!)

(Scratch space. Nothing on this page will be graded!)