EXAM 1

Math 212, 2020 Summer Term 2, Clark Bray.

Name: Solutions Section:______ Student ID:____________

GENERAL RULES

YOU MUST SHOW ALL WORK AND EXPLAIN ALL REASONING TO RECEIVE CREDIT. CLARITY WILL BE CONSIDERED IN GRADING.

No calculators.

All answers must be reasonably simplified.

All of the policies and guidelines on the class webpages are in effect on this exam.

WRITING RULES

Use black pen only. You may use a pencil for initial sketches of diagrams, but the final sketch must be drawn over in black pen and you must wipe all erasure residue from the paper.

DUKE COMMUNITY STANDARD STATEMENT

"I have adhered to the Duke Community Standard in completing this examination."

Signature: ________________________

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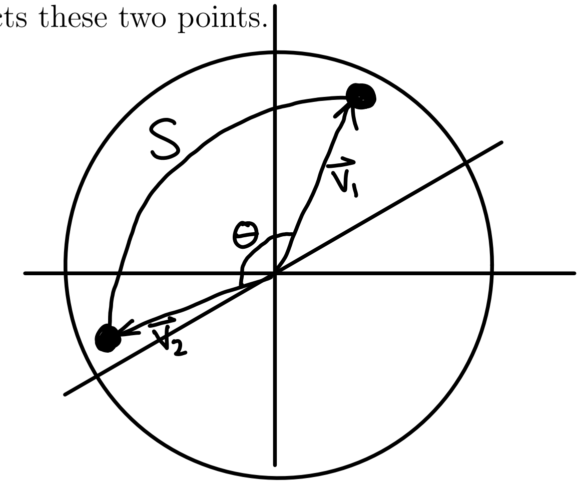
1. *(12 pts)* The points $(3, -6, -2)$ and $(2, 3, 6)$ are on the same sphere centered at the origin. Find the length of the shortest path on this sphere that connects these two points.



$$-24 = \vec{V}_1 \cdot \vec{V}_2$$
$$= \|\vec{V}_1\| \|\vec{V}_2\| \cos\theta$$
$$= (7)(7) \cos\theta$$

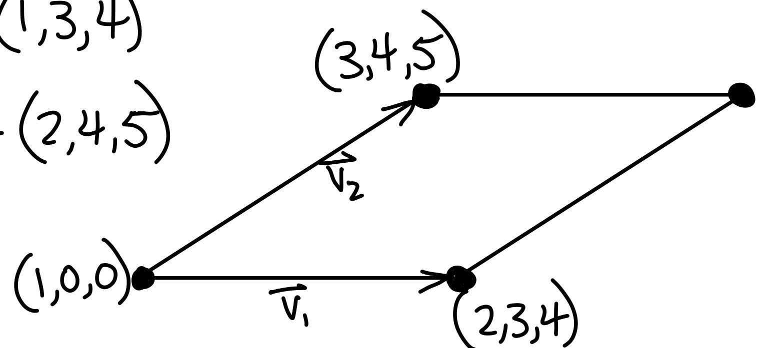
So $\theta = \arccos\left(\frac{-24}{49}\right)$

Then $S = r\theta = 7\arccos\left(\frac{-24}{49}\right)$

2. *(12 pts)* Compute the area of the parallelogram with vertices at $(1, 0, 0)$, $(2, 3, 4)$, $(3, 4, 5)$, $(4, 7, 9)$.

$\vec{V}_1 = (2,3,4) - (1,0,0) = (1,3,4)$

$\vec{V}_2 = (3,4,5) - (1,0,0) = (2,4,5)$



$$\vec{V}_1 \times \vec{V}_2 = \det\begin{pmatrix} \vec{e}_1 & \vec{e}_2 & \vec{e}_3 \\ 1 & 3 & 4 \\ 2 & 4 & 5 \end{pmatrix} = (-1, 3, -2)$$

$$\text{area} = \|\vec{V}_1 \times \vec{V}_2\| = \|(-1,3,-2)\| = \sqrt{14}$$

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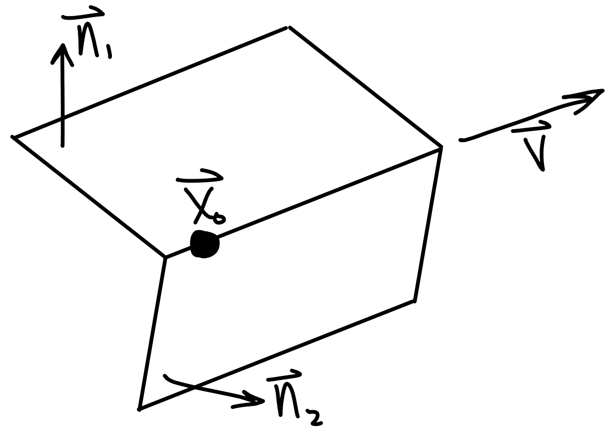
3. *(20 pts)* The planes P_1 and P_2 have equations $x - y - z = -1$ and $x + 2y + 3z = 6$, respectively. Find the symmetric equations for their line of intersection. *(Hint: The point* $(1, 1, 1)$ *might be useful.)*

Both planes include $\vec{x}_0 = (1,1,1)$, which is therefore on the line.



The line is $\perp$ to both $\vec{n}_1, \vec{n}_2$ and thus parallel to

$$\vec{v} = \vec{n}_1 \times \vec{n}_2 = \det\begin{pmatrix} \vec{e}_1 & \vec{e}_2 & \vec{e}_3 \\ 1 & -1 & -1 \\ 1 & 2 & 3 \end{pmatrix} = (-1, -4, 3)$$

So the parametrization is

$$\vec{x} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} + t\begin{pmatrix} -1 \\ -4 \\ 3 \end{pmatrix} \quad \text{or} \quad \begin{aligned} x &= 1 - 1t \\ y &= 1 - 4t \\ z &= 1 + 3t \end{aligned}$$

Solving for t in each of these equations gives us the symmetric equations

$$\frac{x-1}{-1} = \frac{y-1}{-4} = \frac{z-1}{3}$$

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4. *(20 pts)*

(a) The curve C in the xy-plane has equation $y + x^2 y = e^x$. Find a function f whose graph is this curve, and identify the domain and target of f.

Rewriting as $y = \frac{e^x}{1+x^2}$ shows this is the graph $y=f(x)$ of $f:\mathbb{R}^1 \to \mathbb{R}^1$ defined by

$$f(x) = \frac{e^x}{1+x^2}$$

(b) Viewing the xy-plane as sitting inside of xyz-space, the surface S is obtained by rotating C around the x-axis. Find a function g for which one of the level sets is S, and identify the domain and target of g.

S has equation $(1+x^2)\sqrt{y^2+z^2} = e^x$

(because this equation is rotationally symmetric around the x-axis and has the correct cross section in $\{z=0, y \geq 0\}$.)

This is the level set $g=0$ of $g:\mathbb{R}^3 \to \mathbb{R}^1$ defined by

$$g(x,y,z) = (1+x^2)\sqrt{y^2+z^2} - e^x$$

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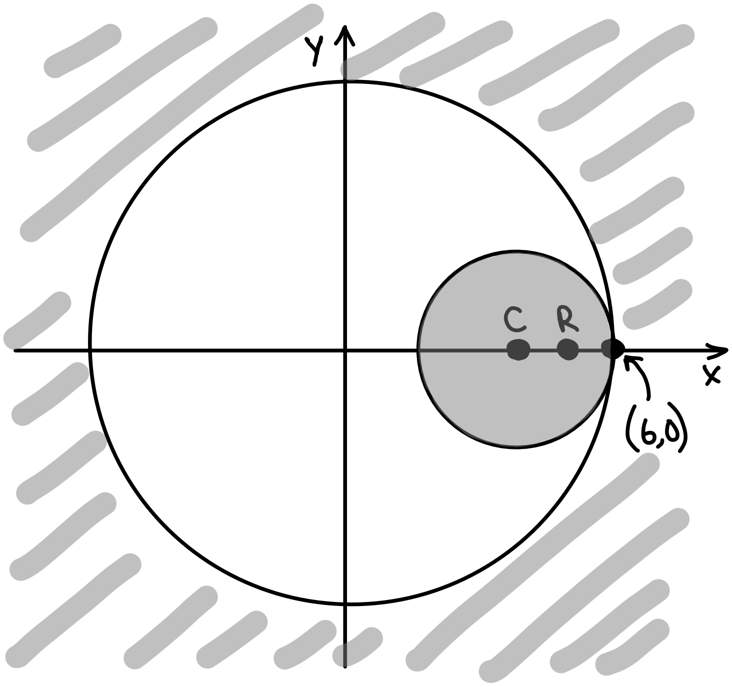
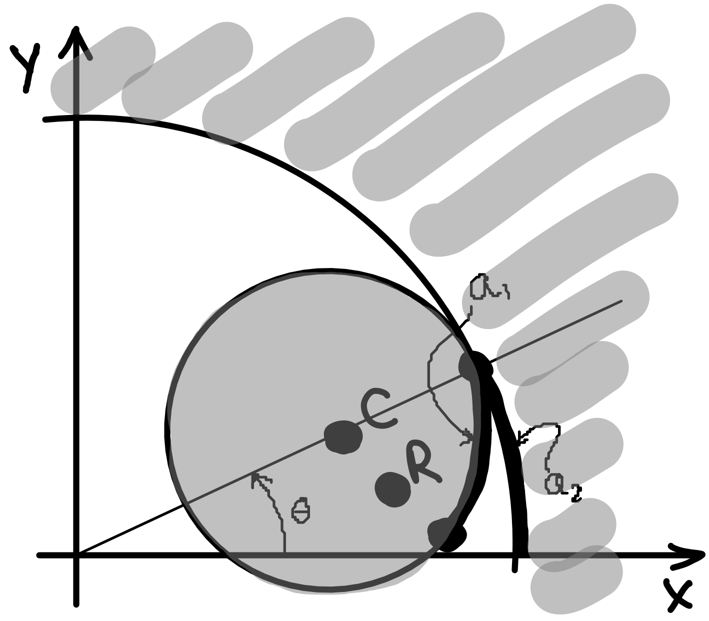
5. *(16 pts)* The xy-plane is made of a sheet of wood. A disk (center at the origin, radius 6) is cut out of the wood and discarded. A new disk of wood (radius 2) is placed with its center C initially at $(4, 0)$ (thus touching the cut circle at $(6, 0)$). There is a red dot R on the new disk, initially at $(5, 0)$.

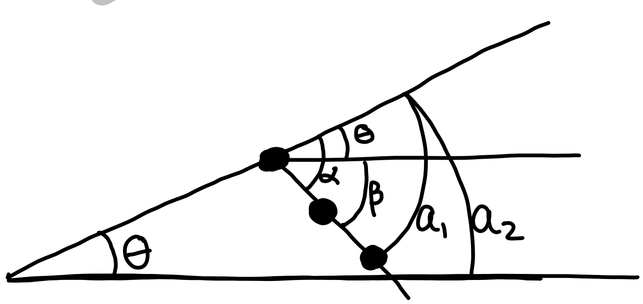
The smaller disk then begins to roll along the outer circle without slipping (in the figure, the arc a_1 must have the same length as the arc a_2), with C moving counterclockwise.

Find a parametrization of the path followed by the red dot R. *(Hint: Use the angle θ at the origin between C and the positive part of the x-axis as your parameter.)*

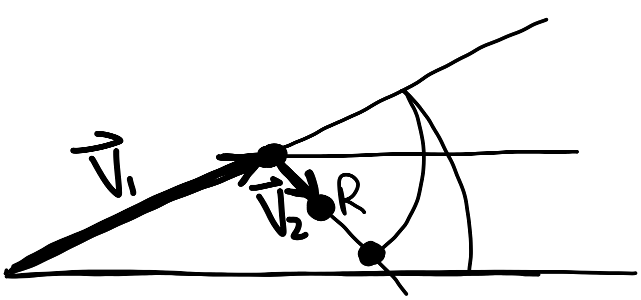




$$a_1 = a_2$$
$$2\alpha = 6\theta$$
$$\alpha = 3\theta$$



$$\beta = \alpha - \theta$$
$$= 2\theta$$



$$\vec{V}_1 = \begin{pmatrix} 4\cos\theta \\ 4\sin\theta \end{pmatrix}$$

$$\vec{V}_2 = \begin{pmatrix} 1\cos\beta \\ -1\sin\beta \end{pmatrix} = \begin{pmatrix} \cos 2\theta \\ -\sin 2\theta \end{pmatrix}$$

$$\vec{R} = \vec{V}_1 + \vec{V}_2 = \begin{pmatrix} 4\cos\theta \\ 4\sin\theta \end{pmatrix} + \begin{pmatrix} \cos 2\theta \\ -\sin 2\theta \end{pmatrix}$$

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6. *(10 pts)* Compute the limit below or show that it does not exist.

$$\lim_{(x,y)\to(0,0)} \frac{xy^2 - x^3 - y^3}{x^2 + y^2}$$

In polar coordinates we get

$$= \lim_{r\to 0} \frac{(r\cos\theta)(r\sin\theta)^2 - (r\cos\theta)^3 - (r\sin\theta)^3}{r^2}$$

$$= \lim_{r\to 0} \underbrace{(r)}_{\text{appoaches } 0!} \underbrace{\left(\cos\theta \sin^2\theta - \cos^3\theta - \sin^3\theta\right)}_{\text{bounded!}}$$

So this limit is zero.

7. *(10 pts)* For the linear transformation $S : \mathbb{R}^2 \to \mathbb{R}^3$, we know $S(1, 2) = (1, 3, 1)$ and $S(0, 1) = (2, 0, 3)$. Find the matrix representing S.

$$\begin{pmatrix}1\\0\end{pmatrix} = \begin{pmatrix}1\\2\end{pmatrix} - 2\begin{pmatrix}0\\1\end{pmatrix}$$

$$S\begin{pmatrix}1\\0\end{pmatrix} = S\begin{pmatrix}1\\2\end{pmatrix} - 2S\begin{pmatrix}0\\1\end{pmatrix} = \begin{pmatrix}1\\3\\1\end{pmatrix} - 2\begin{pmatrix}2\\0\\3\end{pmatrix} = \begin{pmatrix}-3\\3\\-5\end{pmatrix}$$

Then the matrix is

$$A = \begin{pmatrix} S(\vec{e}_1) & S(\vec{e}_2) \end{pmatrix} = \begin{pmatrix} -3 & 2 \\ 3 & 0 \\ -5 & 3 \end{pmatrix}$$

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