EXAM 2

Math 212, 2020 Summer Term 2, Clark Bray.

Name: Solutions Section: ______ Student ID: ____________

GENERAL RULES

YOU MUST SHOW ALL WORK AND EXPLAIN ALL REASONING TO RECEIVE CREDIT. CLARITY WILL BE CONSIDERED IN GRADING.

No calculators.

All answers must be reasonably simplified.

All of the policies and guidelines on the class webpages are in effect on this exam.

WRITING RULES

Use black pen only. You may use a pencil for initial sketches of diagrams, but the final sketch must be drawn over in black pen and you must wipe all erasure residue from the paper.

DUKE COMMUNITY STANDARD STATEMENT

"I have adhered to the Duke Community Standard in completing this examination."

Signature: ________________________

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1. *(24 pts)* In this question we consider the function $f : \mathbb{R}^2 \to \mathbb{R}^2$ defined by $f(x,y) = (xy, \sin(\pi x y^2)) = (f_1, f_2)$ near the point $\vec{a} = (2,1)$ in its domain.

   (a) Compute the directional derivative of f at $\vec{a}$ with velocity $\vec{v} = (3,4)$ *directly from the definition.*

$$D_{\vec{v}} f(\vec{a}) = \frac{d}{dt}\Big|_{t=0} f(\vec{a} + t\vec{v}) = \frac{d}{dt}\Big|_{t=0} f\begin{pmatrix} 2+3t \\ 1+4t \end{pmatrix}$$

$$= \frac{d}{dt}\Big|_{t=0} \left((2+3t)(1+4t),\ \sin\left(\pi(2+3t)(1+4t)^2\right) \right)$$

$$= \frac{d}{dt}\Big|_{t=0} \left(2+11t+12t^2,\ \sin\left(\pi(2+19t+56t^2+48t^3)\right) \right)$$

$$= \left(11+24t,\ \pi(19+112t+144t^2)\cos\left(\pi(2+19t+56t^2+48t^3)\right) \right)\Big|_{t=0}$$

$$= (11,\ 19\pi)$$

   (b) Compute the same directional derivative from part (a) using the derivative transformation.

f is a combination of known differentiable functions, so it is differentiable. Then

$$D_{\vec{v}} f(\vec{a}) = D_{f,\vec{a}}(\vec{v}) = J_{f,\vec{a}}\,\vec{v}$$

$$= \begin{pmatrix} y & x \\ \pi y^2 \cos(\pi x y^2) & 2\pi x y \cos(\pi x y^2) \end{pmatrix}\Bigg|_{(2,1)} \begin{pmatrix} 3 \\ 4 \end{pmatrix} = \begin{pmatrix} 1 & 2 \\ \pi & 4\pi \end{pmatrix}\begin{pmatrix} 3 \\ 4 \end{pmatrix}$$

$$= (11,\ 19\pi)$$

   (c) In what direction from $\vec{a}$ is the component function f_1 increasing the fastest? And, what is the slope of the graph of f_1 in that direction?

∇f_1 is the top row of $J_f = (1,2)$

$$\text{Direction} = \frac{\nabla f_1}{\|\nabla f_1\|} = \frac{(1,2)}{\sqrt{5}}$$

$$\text{slope} = \|\nabla f_1\| = \sqrt{5}$$

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2. *(20 pts)* The quantities E, θ, a, b, x, y, z related to the design of an electric motor are related by the equation

$$E = a^2bx - 3ab^2y + \theta xyz^2$$

(a) In considering one aspect of the motor, with $\theta = \pi$, $a = 1$, $b = 2$, $x = 3$, $y = 4$, $z = 5$ we view x, y, and z as being fixed and the others as variables. Suppose that $\frac{d\theta}{dt} = 6$ and $\frac{da}{dt} = 7$; what should $\frac{db}{dt}$ be in order to cause the rate of change of E to be 8?

$$\frac{dE}{dt} = \frac{\partial E}{\partial \theta}\frac{d\theta}{dt} + \frac{\partial E}{\partial a}\frac{da}{dt} + \frac{\partial E}{\partial b}\frac{db}{dt}$$

$$\frac{dE}{dt} = (xyz^2)\frac{d\theta}{dt} + (2abx - 3b^2y)\frac{da}{dt} + (a^2x - 6aby)\frac{db}{dt}$$

$$8 = (300)6 + (12-48)7 + (3-48)\frac{db}{dt}$$

$$45\frac{db}{dt} = 1800 - 8 - 252 = 1540$$

$$\frac{db}{dt} = \frac{1540}{45} = \frac{308}{9}$$

(b) Now viewing a, x, and y as fixed and the others as variables, suppose we wish to consider the consequences of also fixing E. If we do so, then (using the values noted above) what is $\frac{d\theta}{dz}$?

$$E = a^2bx - 3ab^2y + \theta xyz^2 = F(\theta, b, z)$$

$\frac{\partial F}{\partial \theta} = xyz^2 = 300 \neq 0$, so we can view θ as a function of b, z.

Then taking $\frac{\partial}{\partial z}$ gives us

$$0 = 0 - 0 + \frac{\partial}{\partial z}(\theta)(xyz^2)$$

$$= \left(\frac{\partial \theta}{\partial z}\right)(xyz^2) + (\theta)(2xyz)$$

$$\Rightarrow \frac{\partial \theta}{\partial z} = \frac{-2xyz\theta}{xyz^2} = \frac{-2\pi}{5}$$

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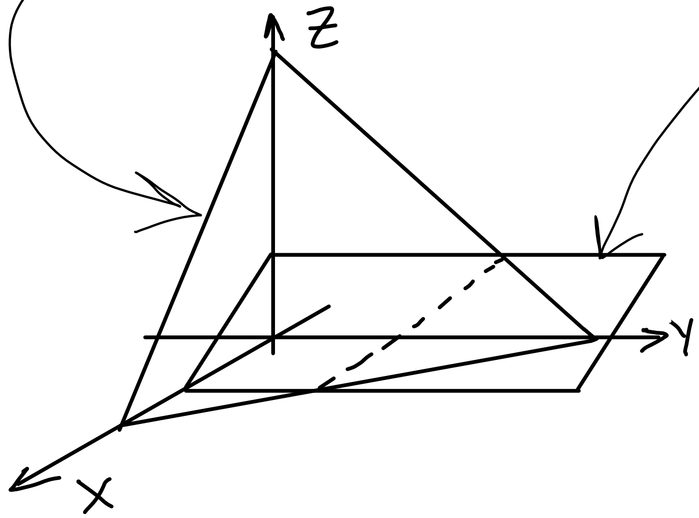
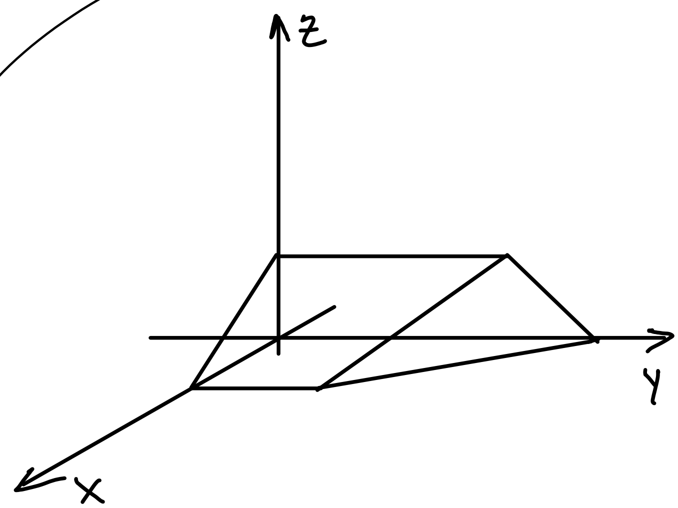
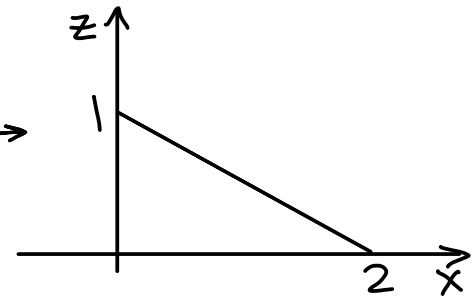
3. *(18 pts)* The domain D is the portion of the first octant (x, y, z all ≥ 0) with $x + 2z \leq 2$ and $3x + 2y + z \leq 12$. Mass is distributed through D with density $\delta(x, y, z) = y$. Set up (but do not evaluate) a triple nested integral representing the mass in D.





$$m = \iiint_D dm = \iiint_D \delta \, dV = \iiint_D y \, dV$$

Projection to the xz-plane is



and for each such point the y bounds are determined from the planes $y=0$ and $3x+2y+z=12$. So we have

$$m = \int_0^1 \int_0^{2-2z} \int_0^{\frac{12-3x-z}{2}} y \, dy \, dx \, dz$$

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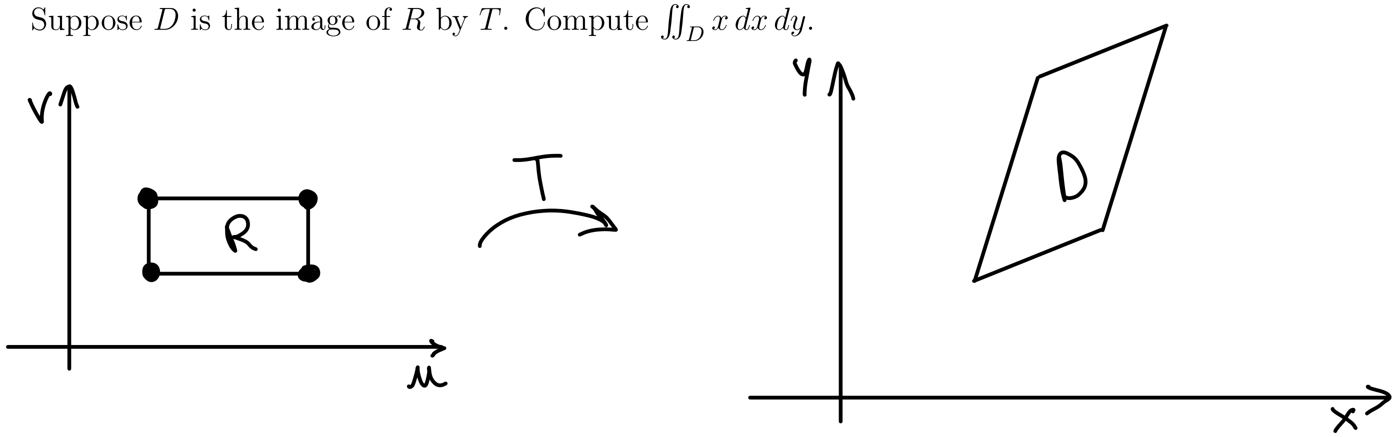
4. *(18 pts)* R is the rectangle with vertices at $(1,1)$, $(3,1)$, $(1,2)$, $(3,2)$, and $T : \mathbb{R}^2 \to \mathbb{R}^2$ is the linear transformation represented by

$$A = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$$

Suppose D is the image of R by T. Compute $\iint_D x\, dx\, dy$.



$$T(u,v) = \begin{pmatrix} u+2v \\ 3u+4v \end{pmatrix} \quad J_T = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} \quad |\det J_T| = |-2| = 2$$

$$\iint_D x \, dx\,dy = \iint_R x \, |\det J_T| \, du\,dv$$

$$= \int_1^2 \int_1^3 (u+2v)(2)\, du\,dv$$

$$= \int_1^2 \left(u^2 + 4uv\right]_{u=1}^{u=3} dv$$

$$= \int_1^2 8 + 8v \, dv$$

$$= 8v + 4v^2 \Big]_1^2 = 20$$

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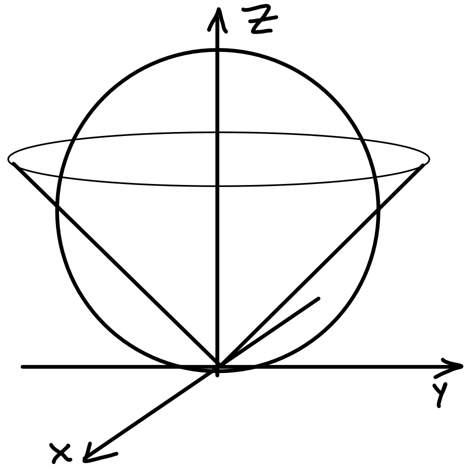
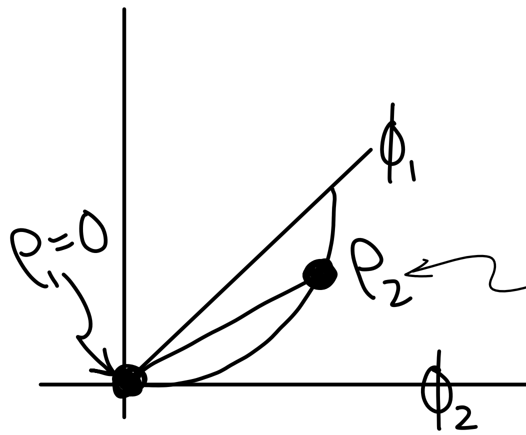
5. *(20 pts)* Let M be the solid defined by $z^2 \le x^2 + y^2$ and $x^2 + y^2 + (z-1)^2 \le 1$.

(a) Set up but do not evaluate a triple nested integral in spherical coordinates representing $\iiint_M x^2yz\,dx\,dy\,dz$.

$$z^2 \le r^2 \iff -r \le z \le r$$





$$\rho^2 - 2z + 1 = 1$$

$$\rho = 2\cos\phi$$

$$\theta \in [0, 2\pi] \qquad \phi \in [\pi/4, \pi/2]$$

$$\iiint_M x^2yz\,dV = \int_0^{2\pi}\int_{\pi/4}^{\pi/2}\int_0^{2\cos\phi} (\rho\sin\phi\cos\theta)^2(\rho\sin\phi\sin\theta)(\rho\cos\phi)\,\rho^2\sin\phi\,d\rho\,d\phi\,d\theta$$

(b) Compute the value of $\iiint_M x^2yz\,dx\,dy\,dz$ using any method from this course.

Reflection through the xz-plane is $R(x,y,z) = (x,-y,z)$

$f(x,y,z) = x^2yz$ has odd symmetry through this plane:

$$f(R(x,y,z)) = f(x,-y,z) = x^2(-y)z$$
$$= -x^2yz = -f(x,y,z)$$

M is symmetric through this same plane.

So $\iiint_M x^2yz\,dx\,dy\,dz = 0$ by symmetry.

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