EXAM 3

Math 212, 2020 Summer Term 2, Clark Bray.

Name: Solutions ______ Section: ______ Student ID: ______

GENERAL RULES

YOU MUST SHOW ALL WORK AND EXPLAIN ALL REASONING TO RECEIVE CREDIT. CLARITY WILL BE CONSIDERED IN GRADING.

No calculators.

All answers must be reasonably simplified.

All of the policies and guidelines on the class webpages are in effect on this exam.

WRITING RULES

Use black pen only. You may use a pencil for initial sketches of diagrams, but the final sketch must be drawn over in black pen and you must wipe all erasure residue from the paper.

DUKE COMMUNITY STANDARD STATEMENT

"I have adhered to the Duke Community Standard in completing this examination."

Signature: ______

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1. *(18 pts)* The curve C is parametrized by $\vec{x}(t) = (t^2, 3t, e^t)$ for $t \in [0,1]$. Compute the line integral along C of the vector field $\vec{F}(x,y,z) = (y, x+z, y)$.

$$\nabla \times \vec{F} = (1-1, 0-0, 1-1) = \vec{0} \quad \Rightarrow \vec{F} = \nabla f$$

$$f = \int y \, dx = xy + c_1(y,z)$$

$$f = \int x+z \, dy = xy + yz + c_2(x,z)$$

$$f = \int y \, dz = yz + c_3(x,y)$$

We can choose $f = xy + yz$

Then
$$\int_C \vec{F} \cdot d\vec{x} = \int_C \nabla f \cdot d\vec{x} = f(\vec{b}) - f(\vec{a})$$

$$= f(1,3,e) - f(0,0,1)$$

$$= (3+3e) - (0+0)$$

$$= 3 + 3e$$

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2. *(18 pts)* Let D be the unit disk in the xy-plane. Use the vector field $\vec{G}(x,y) = (ye^{x^2+y^2-1}, -xe^{x^2+y^2-1})$ to help you in computing the double integral over D of the function $f(x,y) = (1+x^2+y^2)e^{x^2+y^2-1}$.

$$\operatorname{grn} \vec{G} = \frac{\partial}{\partial x}\left(-xe^{x^2+y^2-1}\right) - \frac{\partial}{\partial y}\left(ye^{x^2+y^2-1}\right)$$

$$= (-1)e^{x^2+y^2-1} + (-x)\left(2xe^{x^2+y^2-1}\right)$$
$$- (1)e^{x^2+y^2-1} + (y)\left(2ye^{x^2+y^2-1}\right) = -2f(x,y)$$

$$\iint_D f(x,y)\,dA = -\frac{1}{2}\iint_D \operatorname{grn}\vec{G}\,dA = -\frac{1}{2}\int_{\partial D} \vec{G}\cdot d\vec{x}$$

On ∂D we have $x^2+y^2-1=0$, so this becomes

$$= -\frac{1}{2}\int_{\partial D} \begin{pmatrix} y \\ -x \end{pmatrix} \cdot d\vec{x}$$

We parametrize by $\vec{x}(t) = \begin{pmatrix} \cos t \\ \sin t \end{pmatrix}$, $t \in [0, 2\pi]$, then

$$= -\frac{1}{2}\int_0^{2\pi} \begin{pmatrix} \sin t \\ -\cos t \end{pmatrix} \cdot \begin{pmatrix} -\sin t \\ \cos t \end{pmatrix} dt$$

$$= -\frac{1}{2}\int_0^{2\pi} (-1)\,dt$$

$$= \pi$$

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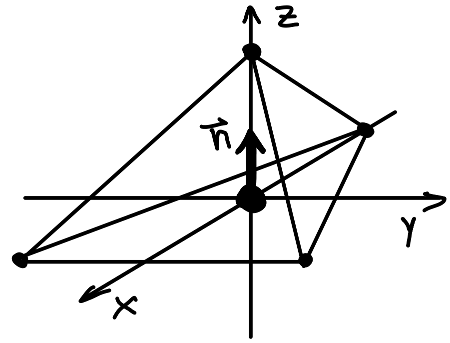
3. *(18 pts)* The surface T is the tetrahedron with vertices at $(1,1,0)$, $(1,-1,0)$, $(-1,0,0)$, $(0,0,1)$, oriented such that at the origin we have $\vec{n} = (0,0,1)$. Compute the flux through T of the vector field $\vec{M}(x,y,z) = (3xy - y^2z, 4x^2z + 5y, xy - z)$.



T is the inward-oriented boundary of the solid tetrahedron R.

So by the divergence theorem we have

$$\iint_T \vec{M} \cdot d\vec{S} = -\iint_{\partial R} \vec{M} \cdot d\vec{S} = -\iiint_R \nabla \cdot \vec{M} \, dV$$

$$= -\iiint_R (3y + 4) \, dV$$

$$= -\underbrace{\iiint 3y \, dV}_{\substack{=0 \text{ by symm.} \\ \text{through xz-plane}}} - 4(\text{vol. of } R)$$

$$= -4\left(\frac{2}{3}\right) = \frac{-8}{3}$$

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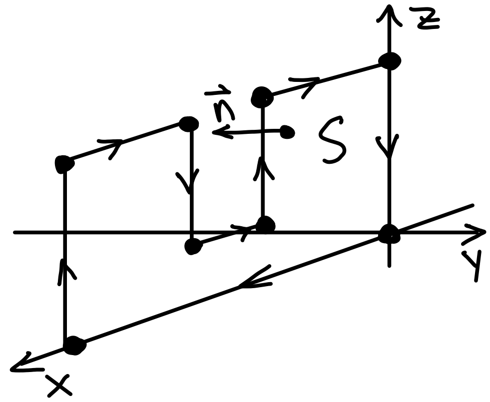
4. *(18 pts)* The curve R in the xz-plane starts at $(0,0,0)$ and then goes to $(5,0,0)$, $(5,0,3)$, $(3,0,3)$, $(3,0,1)$, $(2,0,1)$, $(2,0,3)$, $(0,0,3)$, and then back to the origin. Compute the line integral over R of the vector field $\vec{F}(x,y,z) = (3y - z, 4x + 2z, x - y)$.

R is the boundary of the solid polygon S with orientation $\vec{n} = (0,-1,0)$.



So $\int_R \vec{F} \cdot d\vec{x} = \int_{\partial S} \vec{F} \cdot d\vec{x} = \iint_S (\nabla \times \vec{F}) \cdot d\vec{S}$

$$= \iint_S \begin{pmatrix} -3 \\ -2 \\ 1 \end{pmatrix} \cdot \begin{pmatrix} 0 \\ -1 \\ 0 \end{pmatrix} dS$$

$$= 2 \iint_S dS = 2(\text{area of } S)$$

$$= 26$$

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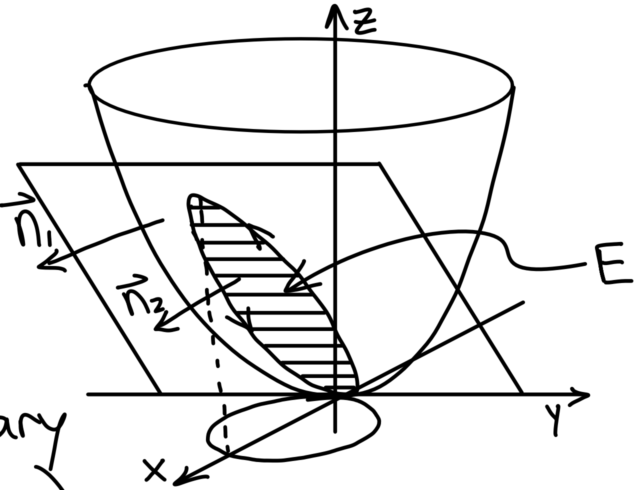
5. *(28 pts)* The surface P is the part of $z = x^2 + y^2$ on or below the plane $z = 2x$, oriented away from the z-axis. Compute the flux through P of $\vec{H}(x, y, z) = (y - 3z, -3x^2z^6e^z, -12x + y)$.

Intersection is $x^2+y^2=2x \Longrightarrow (x-1)^2+y^2=1$

$\nabla\cdot\vec{H} = 0+0+0 = 0$

So $\vec{H}$ is surface independent.



P has the same boundary as E (the intersection curve) oriented as shown in the figure.

$z=2x$

$2x-z=0$

$\vec{n}_2=(2,0,-1)$

$\iint_P \vec{H}\cdot d\vec{S} = \iint_E \vec{H}\cdot d\vec{S}$

Graph parametrization of E is

$(x,y,z)=(u,v,2u)$ over $\{(u-1)^2+v^2\le 1\}$.

$\vec{N}=(1,0,2)\times(0,1,0)=(-2,0,1)$ param backward, so

$$\iint_E \vec{H}\cdot d\vec{S} = -\iint \begin{pmatrix} v-3(2u) \\ -3u^2(2u)^6e^{(2u)} \\ -12u+v \end{pmatrix}\cdot\begin{pmatrix} -2 \\ 0 \\ 1 \end{pmatrix} du\,dv$$

$$= \iint v \, du\,dv = 0$$ by symmetry over the u-axis.

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