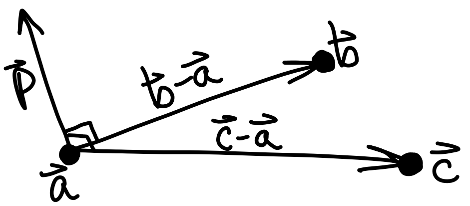
2021 AY Summer 1 Math 212 Exam 1, Solutions

1. (A) $\vec{d} = \vec{k} - \vec{a} = \begin{pmatrix} 3 \\ 3 \\ 6 \end{pmatrix} - \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 2 \\ 3 \\ 6 \end{pmatrix}$

$$\|\vec{d}\| = \sqrt{2^2 + 3^2 + 6^2} = 7$$

(B)



$$\left.\begin{aligned} \vec{b} - \vec{a} &= \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix} \\ \vec{c} - \vec{a} &= \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} \end{aligned}\right\} \text{parallel to } P, \text{ so we choose}$$

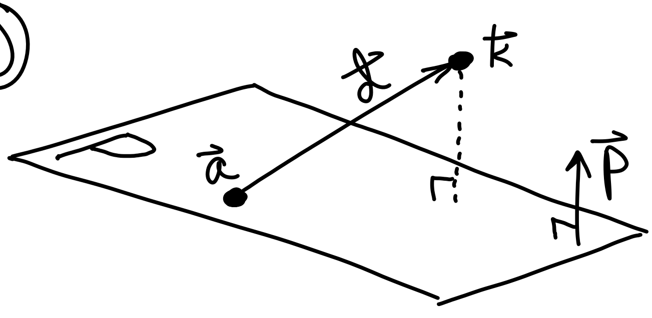
$$\vec{p} = (\vec{b} - \vec{a}) \times (\vec{c} - \vec{a}) = \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix} \times \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$

(C) Q can use the same normal vector $\vec{p}$. Then its equation is

$$\vec{p} \cdot \vec{x} = \vec{p} \cdot \vec{k}$$

$$x + y + z = 12$$

(D)



$$\begin{aligned} \text{distance} &= \operatorname{comp}_{\vec{p}}(\vec{d}) \\ &= \frac{\vec{p} \cdot \vec{d}}{\|\vec{p}\|} \\ &= \frac{(1,1,1) \cdot (2,3,6)}{\sqrt{3}} \\ &= 11/\sqrt{3} \end{aligned}$$

2 A $z^2 = e^y$

$$z^2 = e^{y-3} \qquad \text{"}y\text{"} \mapsto \text{"}y-3\text{"} = \text{shift in pos. } y\text{-direction by } 3.$$

$$z^2 = e^{2y-3} \qquad \text{"}y\text{"} \mapsto \text{"}2y\text{"} = \text{squish in } y\text{-direction by } 2.$$

$$x^2+z^2 = e^{2y-3} \qquad \text{"}z^2\text{"} \mapsto \text{"}x^2+z^2\text{"} = \text{rotate around } y\text{-axis.}$$

$$4x^2+z^2 = e^{2y-3} \qquad \text{"}x\text{"} \mapsto \text{"}2x\text{"} = \text{squish in } x\text{-direction by } 2.$$

B Cross-sections parallel to the xz-plane $(y=c)$ have equation

$$4x^2+z^2 = e^{2c-3} = k > 0$$

so they are ellipses.

③ Ⓐ $xe^y + y^2 z = z^3 \iff x = \dfrac{z^3 - y^2 z}{e^y}$

So S is the graph $x = f(y,z)$ of $f: \mathbb{R}^2 \to \mathbb{R}^1$ defined by

$$f(y,z) = \frac{z^3 - y^2 z}{e^y}$$

$xe^y + y^2 z = z^3 \iff xe^y + y^2 z - z^3 = 0$

So S is the level set $g=0$ of $g: \mathbb{R}^3 \to \mathbb{R}^1$ defined by

$$g(x,y,z) = xe^y + y^2 z - z^3$$

Using the graph parametrization, S is the image of $h: \mathbb{R}^2 \to \mathbb{R}^3$ defined by

$$h(u,v) = \left(\frac{v^3 - u^2 v}{e^u}, u, v \right)$$

Ⓑ From the cylinder we choose $y = 3\cos t$, $z = 3\sin t$.

Then from S we choose

$$x = f(y,z) = \frac{27\sin^3 t - 27\cos^2 t \sin t}{e^{3\cos t}}$$

So we have

$$\vec{x}(t) = \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} \dfrac{27\sin^3 t - 27\cos^2 t \sin t}{e^{3\cos t}} \\ 3\cos t \\ 3\sin t \end{pmatrix}$$

4 A $\vec{v}_1 = (1,4,2)$ $\quad \vec{v}_2 = (0,1,2)$ $\quad \vec{v}_3 = (0,0,1)$

$\vec{e}_1 = \vec{v}_1 - 4\vec{v}_2 + 6\vec{v}_3$ $\quad \vec{e}_2 = \vec{v}_2 - 2\vec{v}_3$ $\quad \vec{e}_3 = \vec{v}_3$

$$T(\vec{e}_1) = T(\vec{v}_1) - 4T(\vec{v}_2) + 6T(\vec{v}_3) = \begin{pmatrix} 3 \\ 2 \\ 5 \end{pmatrix} - 4\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} + 6\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 5 \\ 8 \\ 7 \end{pmatrix}$$

$$T(\vec{e}_2) = T(\vec{v}_2) - 2T(\vec{v}_3) = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} - 2\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} -1 \\ -2 \\ -1 \end{pmatrix}$$

$$T(\vec{e}_3) = T(\vec{v}_3) = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$

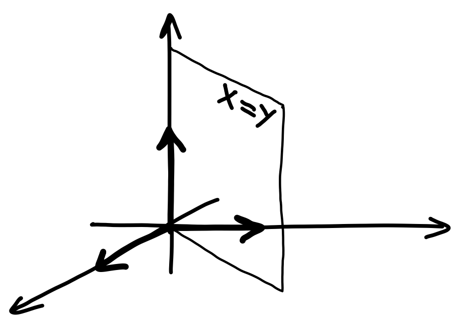
The matrix representing T is

$$A = \begin{pmatrix} T(\vec{e}_1) & T(\vec{e}_2) & T(\vec{e}_3) \end{pmatrix} = \begin{pmatrix} 5 & -1 & 1 \\ 8 & -2 & 1 \\ 7 & -1 & 1 \end{pmatrix}$$

B $S(\vec{e}_1) = \vec{e}_2$

$S(\vec{e}_2) = \vec{e}_1$

$S(\vec{e}_3) = \vec{e}_3$



So S is represented by

$$B = \begin{pmatrix} S(\vec{e}_1) & S(\vec{e}_2) & S(\vec{e}_3) \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Then $T \circ S$ is represented by

$$AB = \begin{pmatrix} 5 & -1 & 1 \\ 8 & -2 & 1 \\ 7 & -1 & 1 \end{pmatrix}\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} -1 & 5 & 1 \\ -2 & 8 & 1 \\ -1 & 7 & 1 \end{pmatrix}.$$

⑤ $\vec{a} + t\vec{v} = \begin{pmatrix} 2t \\ 3t \end{pmatrix}$ $\quad g(\vec{a}+t\vec{v}) = \begin{pmatrix} (2t)e^{(3t)} \\ (3t)\sin(2t) \end{pmatrix}$

$$\frac{d}{dt} g(\vec{a}+t\vec{v}) = \begin{pmatrix} (2t)'(e^{3t}) + (2t)(e^{3t})' \\ (3t)'(\sin 2t) + (3t)(\sin 2t)' \end{pmatrix}$$

$$= \begin{pmatrix} (2+6t)e^{3t} \\ 3\sin 2t + 6t\cos 2t \end{pmatrix}$$

$$D_{\vec{v}}\, g(\vec{a}) = \left.\frac{d}{dt}\right|_{t=0} g(\vec{a}+t\vec{v}) = \begin{pmatrix} 2 \\ 0 \end{pmatrix}.$$