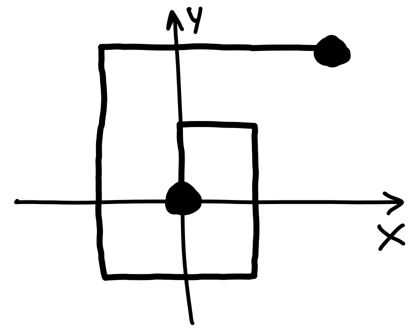
2021 AY Summer 1 Math 212 Exam 3, Solutions

① $$\operatorname{grn}\vec{F} = \frac{\partial}{\partial x}\left(xe^{y} - xe^{xy}\right) - \frac{\partial}{\partial y}\left(e^{y} - ye^{xy}\right)$$



$$= \left(e^{y} - \left(e^{xy} + xye^{xy}\right)\right) - \left(e^{y} - \left(e^{xy} + xye^{xy}\right)\right)$$

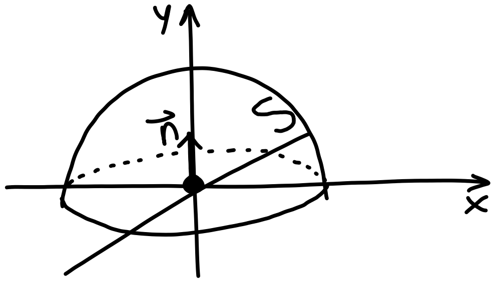
$$= 0 \implies \vec{F} = \nabla f \text{ for some } f.$$

$$\left.\begin{aligned} f &= \int e^{y} - ye^{xy}\,dx = xe^{y} - e^{xy} + c_1(y) \\ f &= \int xe^{y} - xe^{xy}\,dy = xe^{y} - e^{xy} + c_2(x) \end{aligned}\right\} \text{Can choose } f = xe^{y} - e^{xy}$$

F.T.L.I. then gives us

$$\int_C \vec{F} \cdot d\vec{x} = f(\vec{b}) - f(\vec{a}) = f(2,2) - f(0,0)$$

$$= 2e^{2} - e^{4} + 1$$

② 

Let B be the upper half of the unit ball. Then $S = -\partial B$, because S is oriented inward.

By the divergence theorem,

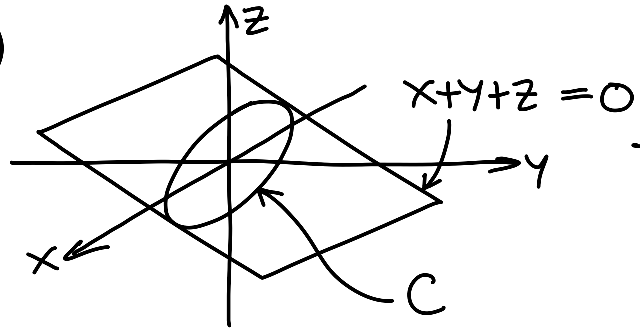
$$\iint_S \vec{G} \cdot d\vec{S} = -\iint_{\partial B} \vec{G} \cdot d\vec{S} = -\iiint_B \nabla \cdot \vec{G} \, dV$$

$$= -\iiint_B \left(2y^3 + 3\right) dV$$

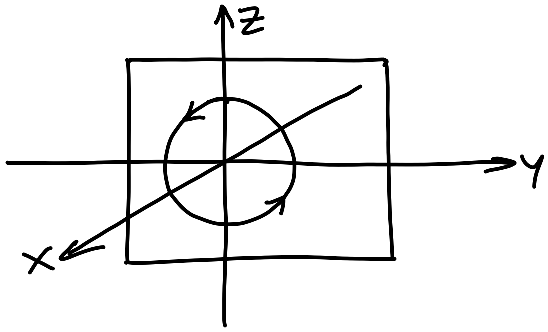
$$= \underbrace{-2\iiint_B y^3 \, dV}_{\substack{=0 \text{ by symmetry} \\ \text{through the} \\ xz\text{-plane}}} - 3\underbrace{\iiint_B dV}_{\substack{= \frac{1}{2}(\text{volume of the} \\ \text{unit sphere})}}$$

$$= -2\pi$$

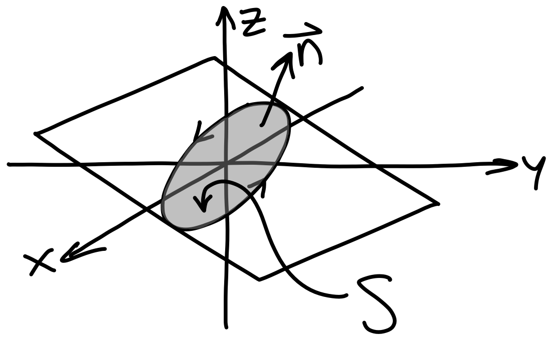
③



$\vec{x}$ is on the plane $x+y+z=0$



and its projection to the yz-plane is a circle oriented CCW as seen from the positive x-axis.



So C is the boundary of the solid ellipse, with $\vec{n}$ having a positive x-coordinate, thus $\vec{n} = (1,1,1)/\sqrt{3}$.

Stokes's theorem then gives us

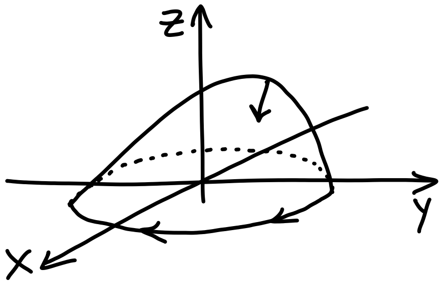
$$\int_C \vec{H}\cdot d\vec{x} = \int_{\partial S} \vec{H}\cdot d\vec{x}$$

$$= \iint_S (\nabla\times\vec{H})\cdot\vec{n}\, dS$$

$$= \iint_S \begin{pmatrix}2\\-1\\-1\end{pmatrix}\cdot\begin{pmatrix}1\\1\\1\end{pmatrix}/\sqrt{3}\, dS$$

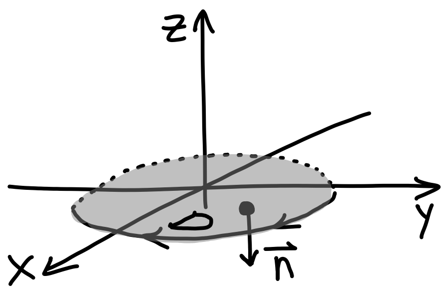
$$= \iint_S 0\, dS = 0$$

④ $\nabla \cdot \vec{F} = (-e^x) + (0) + (e^x) = 0$, so $\vec{F}$ is surface independent.



The boundary of S is where $(1-x^2-y^2)e^{x+y} = 0 \iff x^2+y^2 = 1$

So ∂S is the unit circle in the xy-plane. S is oriented downward so ∂S is oriented clockwise as seen from above.



S then has the same boundary as the unit disk D oriented downward.

So
$$\iint_S \vec{F}\cdot d\vec{S} = \iint_D \vec{F}\cdot d\vec{S} = \iint_D \vec{F}\cdot\vec{n}\, dS$$

$$= \iint_D \vec{F}\cdot\begin{pmatrix}0\\0\\-1\end{pmatrix} dS = \iint_D (-ze^x + x^3 - 5)\, dS$$

$$= \underbrace{\iint_D -ze^x\, dS}_{=0 \text{ b/c } z=0 \text{ on } D} + \underbrace{\iint_D x^3\, dS}_{=0 \text{ by symm through the } yz\text{-plane}} - \underbrace{\iint_D 5\, dS}_{=-5(\text{area of } D)}$$

$$= -5\pi$$

⑤ $f = x^2 - y$ is differentiable everywhere and

$$\nabla f = \begin{pmatrix} 2x \\ -1 \end{pmatrix}$$ is never $\vec{0}$, so there are no interior critical points.

On the boundary where $g(x,y) = x^2 + y^2 = 1$, we again have f differentiable; and

$$\nabla g = \begin{pmatrix} 2x \\ 2y \end{pmatrix}$$

is $\vec{0}$ only at the origin which is not on that boundary. The Lagrange condition gives us

$$\nabla f = \lambda \nabla g \iff \begin{pmatrix} 2x \\ -1 \end{pmatrix} = \lambda \begin{pmatrix} 2x \\ 2y \end{pmatrix}$$

$$\iff \begin{aligned} 2x &= \lambda(2x) & \quad (1) \\ -1 &= \lambda(2y) & \quad (2) \end{aligned}$$

Case 1: $x = 0$, in which case we get $y = \pm 1$.

Case 2: $x \neq 0$, so (1) $\Rightarrow \lambda = 1$, then (2) $\Rightarrow y = -1/2$ and thus $x = \pm\sqrt{3}/2$.

Check values:

$f(0,1) = -1$ $\quad f\left(\frac{\sqrt{3}}{2}, \frac{-1}{2}\right) = 5/4$ $\quad$ Min.

$f(0,-1) = 1$ $\quad f\left(\frac{-\sqrt{3}}{2}, \frac{-1}{2}\right) = 5/4$ $\quad$ Max.