2021 AY Summer 2 Math 212 Exam 2, Solutions

① Ⓐ $\frac{\partial f_1}{\partial x} = y^3 \quad \frac{\partial f_1}{\partial y} = 3xy^2$ These partials are all continuous.

$\frac{\partial f_2}{\partial x} = ye^x \quad \frac{\partial f_2}{\partial y} = e^x$ So f is continuously differentiable, and thus also differentiable.

$\frac{\partial f_3}{\partial x} = 1 \quad \frac{\partial f_3}{\partial y} = 1$

Ⓑ
$$J_f = \begin{pmatrix} y^3 & 3xy^2 \\ ye^x & e^x \\ 1 & 1 \end{pmatrix} \qquad J_{f,\vec{a}} = \begin{pmatrix} 8 & 0 \\ 2 & 1 \\ 1 & 1 \end{pmatrix}$$

$$D_{\vec{v}} f(\vec{a}) = J_{f,\vec{a}} \vec{v} = \begin{pmatrix} 8 & 0 \\ 2 & 1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 3 \end{pmatrix} = \begin{pmatrix} 8 \\ 5 \\ 4 \end{pmatrix}$$

Ⓒ Direction is that of $\nabla f_2 = \begin{pmatrix} \partial f_2/\partial x \\ \partial f_2/\partial y \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$,

which is $\vec{u} = \begin{pmatrix} 2 \\ 1 \end{pmatrix} / \sqrt{5}$.

Steepness is $\|\nabla f_2\| = \sqrt{5}$.

② Ⓐ To avoid confusion of variables, we rewrite as

$$\begin{pmatrix} x \\ y \end{pmatrix} \xrightarrow{f} \begin{pmatrix} u \\ v \end{pmatrix} \xrightarrow{g} \begin{pmatrix} h_1 \\ h_2 \\ h_3 \end{pmatrix}$$

$$f\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} xy \\ x+y \end{pmatrix} = \begin{pmatrix} u \\ v \end{pmatrix}, \quad g\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} 2uv \\ u^2 - uv \\ u^3 \end{pmatrix} = \begin{pmatrix} h_1 \\ h_2 \\ h_3 \end{pmatrix}$$

Then $J_f = \begin{pmatrix} y & x \\ 1 & 1 \end{pmatrix}$, $J_g = \begin{pmatrix} 2v & 2u \\ 2u-v & -u \\ 3u^2 & 0 \end{pmatrix}$

And $J_h = J_{g \circ f} = J_g J_f$

$$= \begin{pmatrix} 2v & 2u \\ 2u-v & -u \\ 3u^2 & 0 \end{pmatrix} \begin{pmatrix} y & x \\ 1 & 1 \end{pmatrix}$$

At $x=1, y=2$ we have $u=2, v=3$, so the above becomes

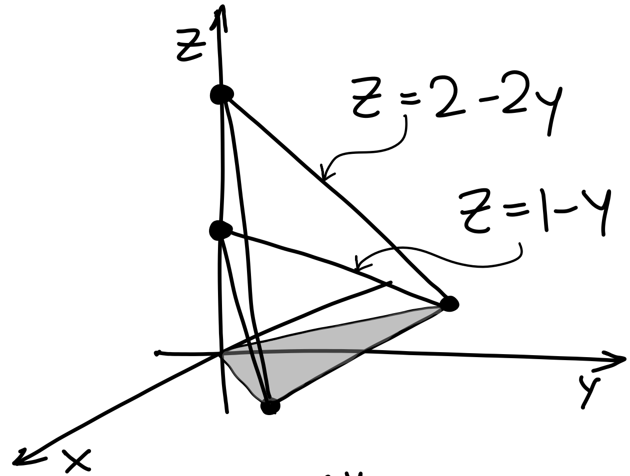
$$J_{h,(1,2)} = \begin{pmatrix} 6 & 4 \\ 1 & -2 \\ 12 & 0 \end{pmatrix} \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} 16 & 10 \\ 0 & -1 \\ 24 & 12 \end{pmatrix}$$

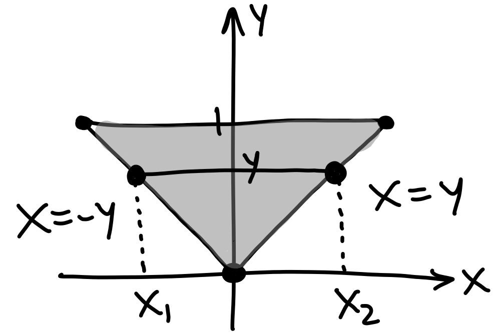
Ⓑ $\frac{\partial h_2}{\partial y} = -1 \neq 0$, so yes $h_2 = c$ does allow for viewing y as a function of x near $(1,2)$.

③

$$m = \iiint_R \delta \, dV$$

The projection of R to the xy-plane is the shaded triangle as shown.





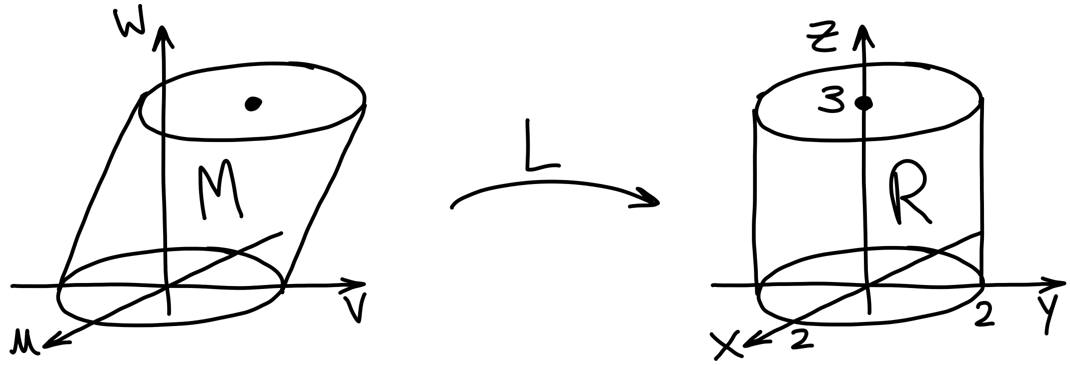
So we choose to order the differentials "dz dx dy".

Then

$$m = \iiint_R \delta \, dV$$

$$= \int_0^1 \int_{-y}^{y} \int_{1-y}^{2-2y} 1 + \sin(xyz) \; dz \, dx \, dy$$

4



(A) $\det(J_L) = \dfrac{\partial(x,y,z)}{\partial(u,v,w)} = \det\begin{pmatrix} 1 & 0 & -1/3 \\ 0 & 1 & -2/3 \\ 0 & 0 & 1 \end{pmatrix} = 1.$

(B) $\iiint_M u \, du\,dv\,dw = \iiint_R u \left| \dfrac{\partial(u,v,w)}{\partial(x,y,z)} \right| dx\,dy\,dz$

$= \iiint_R u \left(\left| \dfrac{\partial(x,y,z)}{\partial(u,v,w)} \right| \right)^{-1} dx\,dy\,dz = \iiint_R u \, dx\,dy\,dz$

$= \iiint_R \left(x + \frac{1}{3}z\right) dx\,dy\,dz = \iiint_R x \, dx\,dy\,dz + \iiint_R \frac{1}{3} z \, dx\,dy\,dz$

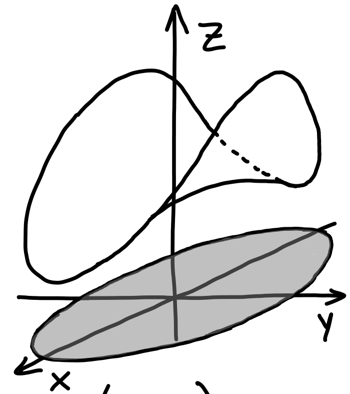
(R is symmetric through the yz-plane. The reflection F is $F(x,y,z) = (-x,y,z)$, so $g(x,y,z) = x$ is odd because $g(F(x,y,z)) = g(-x,y,z) = -x = -g(x,y,z)$. So this integral is zero.)

$= \iiint_R \frac{1}{3} z \, dx\,dy\,dz = \frac{1}{3} \int_0^{2\pi} \int_0^2 \int_0^3 z \, r \, dz\,dr\,d\theta$

$= \frac{1}{3} \cdot \frac{9}{2} \int_0^{2\pi} \int_0^2 r \, dr\,d\theta = \frac{1}{3} \cdot \frac{9}{2} \cdot 2 \int_0^{2\pi} d\theta = \frac{1}{3} \cdot \frac{9}{2} \cdot 2 \cdot 2\pi$

$= 6\pi$

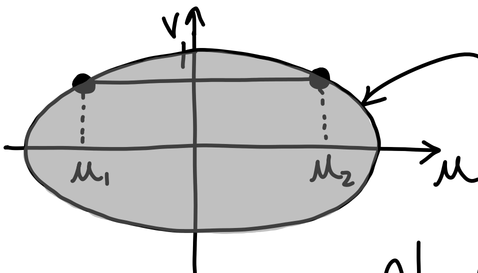
5 A We parametrize H by



$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} u \\ v \\ 6+2v^2-u^2 \end{pmatrix}$$

Then

$$\vec{X}_u = \begin{pmatrix} 1 \\ 0 \\ -2u \end{pmatrix} \quad \vec{X}_v = \begin{pmatrix} 0 \\ 1 \\ 4v \end{pmatrix} \quad \vec{N} = \begin{pmatrix} 2u \\ -4v \\ 1 \end{pmatrix}$$

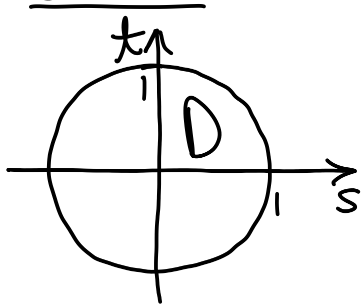


$u^2+4v^2=4$

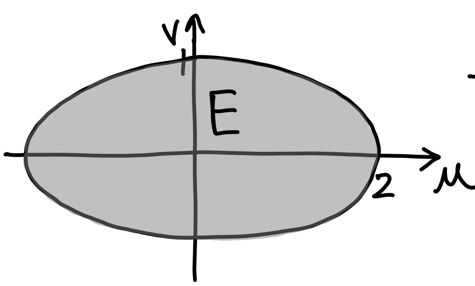
$v_1=-1,\ v_2=1$; $u_1 = -\sqrt{4-4v^2}$, $u_2 = \sqrt{4-4v^2}$

$$\text{area} = \int_{-1}^{1}\int_{-\sqrt{4-4v^2}}^{\sqrt{4-4v^2}} \sqrt{1+(2u)^2+(-4v)^2}\ du\,dv$$

<u>Bonus:</u>



$\xrightarrow[t=v]{2s=u}$



$$\frac{\partial(u,v)}{\partial(s,t)} = \det\begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix} = 2$$

$$\iint_D \sqrt{1+(2(2s))^2+(-4v)^2}\,(2\,ds\,dt) = \iint_E \sqrt{1+(2u)^2+(-4v)^2}\ du\,dv$$

$\|$ (polar coords in st-plane)

$$\int_0^{2\pi}\int_0^1 \sqrt{1+16r^2}\ 2r\,dr\,d\theta$$

Let $f = 1+16r^2$
$\Rightarrow df = 32r\,dr$

$$= \frac{1}{16}\int_0^{2\pi}\int \sqrt{t}\, dt\, d\theta = \frac{1}{16}\int_0^{2\pi}\left(\frac{2}{3} t^{3/2}\right] d\theta$$

$$= \frac{1}{16}\int_0^{2\pi} \frac{2}{3}\left(1+16r^2\right)^{3/2}\Big]_0^1 d\theta = \frac{1}{16}\int_0^{2\pi}\frac{2}{3}\left(17^{3/2}-1\right) d\theta$$

$$= \frac{\pi}{12}\left(17^{3/2}-1\right)$$