

7.

# Chapter 2 Matrix Algebra $\mathbb{R}$ could be any field

$$\left. \begin{array}{l} \text{matrix } A \in \mathbb{R}^{m \times n} \\ \text{vector } x \in \mathbb{R}_{\text{col}}^n \end{array} \right\} \Rightarrow Ax \in \mathbb{R}_{\text{col}}^m$$

range  
or target

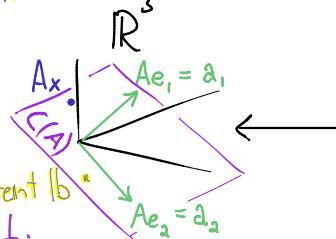
$A \rightsquigarrow$  function  $\mu_A: \mathbb{R}^n \rightarrow \mathbb{R}^m$

inconsistent  $\|b\|$

E.g.  $3 \times 2$

$$\mu_A(e_j) = Ae_j = a_j$$

consistent  $\|b\|$



generically inconsistent:

$b \notin \text{span}(a_1, a_2)$

"Pf": row reduce A

$\rightsquigarrow$  row of 0's

lin.combin.of

$a_1$  and  $a_2$   
with coeffs  $x_1, x_2$

$Ax = b$  has unique sol when...?  $\Leftrightarrow$

- $Ax = b$  is consistent ( $b \in C(A)$ ) and
- no collapsing from  $\mathbb{R}^n$  to  $\mathbb{R}^m$ :  $x \neq y \Rightarrow Ax \neq Ay$

$$\dim(\text{source}) = \dim(\text{image}) \\ n = \text{rank } A$$

E.g.  $3 \times 2$

$$\begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} = \begin{bmatrix} -A_1 \\ -A_2 \\ -A_3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

generically inconsistent:

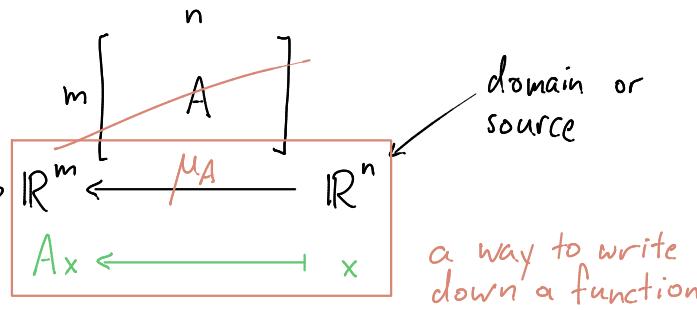
- $x \notin H_1 \cap H_2 \cap H_3$  in  $\mathbb{R}^2$

Def: A function  $T: \mathbb{R}^n \rightarrow \mathbb{R}^m$  is linear (or a linear transformation)

$$\text{if } T(x+cy) = Tx + cTy$$

different "+"!

" operator  
map



A is a matrix;  
 $M_A$  is a function

image of  $\mu_A$  is  $\{\mu_A(x) \mid x \in \text{domain}\} \subseteq \text{range}$

$= M_A(\text{domain})$

$= M_A(\mathbb{R}^n) = \{Ax \mid x \in \mathbb{R}^n\} \subseteq \mathbb{R}^m$

$= C(A)$ .

image of  $\mu_A$  could be

•  $\emptyset$

$\square$

rank: 0 1 2

Note: don't need  
to know rank!  
 $\emptyset$  or  $\square$

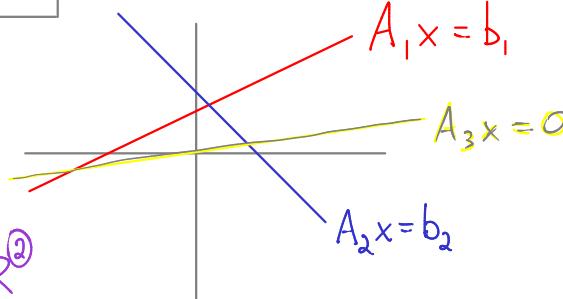
$Ax = b$  consistent when...?  
 $\Leftrightarrow b \in \text{image of } M_A$ .

$Ax = b$  consistent  $\forall b \Leftrightarrow C(A) = \mathbb{R}^m$   
 $\Leftrightarrow \text{rank } A = m$ .

Def:  $\mu_A$  is surjective (onto)

Def:  $\mu_A$  is injective (into or one-to-one)

Think in  $\mathbb{R}^n$  now:

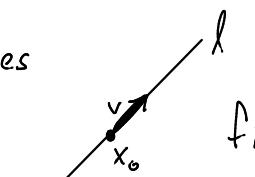


higher dim:  
 $m$  hyperplanes  
in  $\mathbb{R}^n$

E.g. Lemma, last time:  $\mu_A$  is linear.

Geometrically: linear  $\Rightarrow$  takes lines to lines

$$\ell = \{x_0 + cv \mid c \in \mathbb{R}\}$$



fixed  $x_0$  and  $v$

$$T(\ell) = \left\{ \underbrace{T(x_0)}_{y_0} + \underbrace{cT(v)}_u \mid c \in \mathbb{R} \right\}$$

fixed  $y_0$  and  $u$

Linear Algebra!

Prop: Linear maps preserve linear combinations.

Pf: Assume  $T$  is linear. Then

$$\begin{aligned} T(c_1x_1 + \dots + c_kx_k) &= T(c_1x_1 + \dots + c_{k-1}x_{k-1}) + c_k T x_k \quad \text{by linearity} \\ &= c_1 T x_1 + \dots + c_{k-1} T x_{k-1} + c_k T x_k \quad \text{by induction on } k \quad (!) \\ &= \sum_{i=1}^k c_i T x_i. \end{aligned}$$

$$k=1: T(c_1v_1) = \underbrace{T(0)}_{T(0)} + c_1 T(v_1) = 0 + c_1 T v_1 = c_1 T v_1 \quad \checkmark$$

$$T(0) = T(0+0) = T(0) + 1 T(0) \Rightarrow 0 = 1 T(0) = T(0). \quad \square$$

Q.  $\mu_A \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} = ? \quad \alpha_1 = \text{left column of } A$

$$\mu_A(e_1) = \alpha_1$$

$$e_j = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \leftarrow \text{row } j \Rightarrow \boxed{\mu_A(e_j) = \alpha_j}$$

So what? Combine with Prop!

Q. Given linear  $T: \mathbb{R}^n \rightarrow \mathbb{R}^m$ , what can it look like?

A.  $x \in \mathbb{R}^n \Rightarrow x \in \text{span}(e_1, \dots, e_n)$

$\Rightarrow Tx$  determined by what it does to  $e_1, \dots, e_n$ !

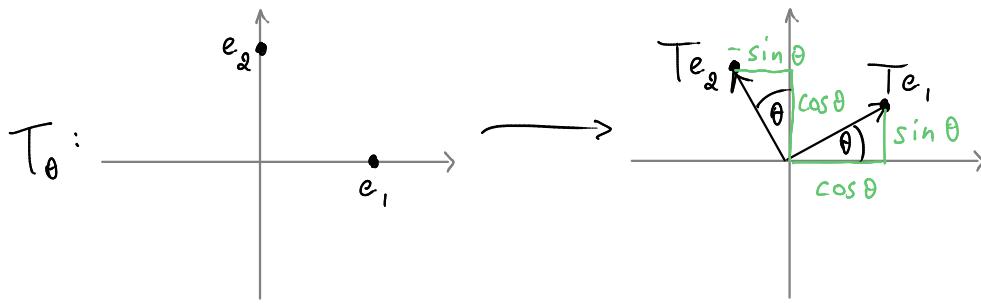
Say  $\alpha_1 = T e_1, \dots, \alpha_n = T(e_n)$ . What's  $Tx$ ?

$x = x_1 e_1 + \dots + x_n e_n$ , so Prop  $\Rightarrow$

$$Tx = x_1 \alpha_1 + \dots + x_n \alpha_n = \mu_A(x) \text{ for } A = \begin{bmatrix} 1 & \dots & 1 \\ T e_1 & \dots & T e_n \\ 1 & \dots & 1 \end{bmatrix}.$$

Hence every linear  $T: \mathbb{R}^n \rightarrow \mathbb{R}^m$  is  $\mu_A$  for some  $A \in \mathbb{R}^{m \times n}$ !

E.g. rotation of  $\mathbb{R}^2$  by angle  $\theta$



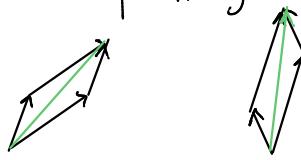
$$\text{Note: } \overline{T_\theta}(cx) = c\overline{T_\theta}(x)$$

rotation preserves circles around 0

$$T_\theta(x+y) = \overline{T_\theta}(x) + \overline{T_\theta}(y)$$

takes parallelograms to congruent parallelograms

$$T_\theta \begin{pmatrix} 1 \\ 0 \end{pmatrix} = ? \quad \begin{bmatrix} \cos \theta \\ \sin \theta \end{bmatrix}$$



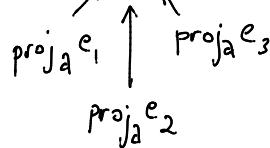
no need to memorize; can reconstruct by picture

$$T_\theta \begin{pmatrix} 0 \\ 1 \end{pmatrix} = ? \quad \begin{bmatrix} -\sin \theta \\ \cos \theta \end{bmatrix} \Rightarrow \overline{T_\theta} = M_A \text{ for } A = \begin{bmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{bmatrix}$$

E.g.  $\text{proj}_{[1]} : \mathbb{R}^3 \rightarrow \mathbb{R}^3$ . Set  $a = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ .  $\text{proj}_a x = \frac{x \cdot a}{\|a\|^2} a = \frac{x_1 + x_2 + x_3}{3} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$

is linear as a function of  $x \Rightarrow \text{proj}_a = M_C$  for  $C = \frac{1}{3} \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$ .

E.g.  $I_n = \begin{bmatrix} 1 & & 0 \\ & \ddots & \\ 0 & & 1 \end{bmatrix} \Rightarrow M_{I_n} : \mathbb{R}^n \rightarrow \mathbb{R}^n$  is ...?



identity map

E.g.  $A = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \Rightarrow M_A = ?$  reflection across x-axis

E.g.  $A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \Rightarrow M_A = ?$  shear

