

19. $T: V \rightarrow W$ linear

$\mathcal{V} = (v_1, \dots, v_n)$ ordered basis of V

$\mathcal{W} = (w_1, \dots, w_m)$ ordered basis of W

compact notation
not in textbook!

$$T[v_1, \dots, v_n] = [w_1, \dots, w_m] [T]_{\mathcal{V}, \mathcal{W}}$$

↑
1x1 symbols

defines $[T]_{\mathcal{V}, \mathcal{W}} = A$

$$Tv_j = a_{1j}w_1 + \dots + a_{mj}w_m = [w_1, \dots, w_m] \begin{bmatrix} a_{1j} \\ \vdots \\ a_{mj} \end{bmatrix}$$

Q. Set $A = [T]_{\mathcal{V}, \mathcal{W}} \in \mathbb{R}^{m \times n}$. What does it mean that $Ax = b$?

A. $T[v_1, \dots, v_n] x = [w_1, \dots, w_m] Ax$

(*) The coefficients x of $v \in V$ on v_1, \dots, v_n get taken to the coefficients b of $w \in W$ on w_1, \dots, w_m

E.g. $D: P_3 \rightarrow P_2$ $\mathcal{V} = (1, t, t^2, t^3)$ and $\mathcal{W} = (1, t, t^2) \Rightarrow [D]_{\mathcal{V}, \mathcal{W}} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 3 \end{bmatrix}$

$$v = x_0 + x_1 t + x_2 t^2 + x_3 t^3 \Rightarrow Dv = x_1 + 2x_2 t + 3x_3 t^2$$

$$\leftrightarrow \begin{bmatrix} x_0 \\ x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

$$\leftrightarrow \begin{bmatrix} b_0 \\ b_1 \\ b_2 \end{bmatrix} = [D]_{\mathcal{V}, \mathcal{W}} x = \begin{bmatrix} x_1 \\ 2x_2 \\ 3x_3 \end{bmatrix}$$

E.g. $V = \mathbb{R}^n$ $W = \mathbb{R}^m$

$$\mathcal{V} = \mathcal{E}_n \quad \mathcal{W} = \mathcal{E}_m$$

$$= e_1, \dots, e_n \quad = e_1, \dots, e_m$$

$$T = M_A \quad m \boxed{n} \quad \Rightarrow [T]_{\mathcal{E}_n, \mathcal{E}_m} = ?$$

$$M_A[e_1, \dots, e_n] = [e_1, \dots, e_m] [T]_{\mathcal{E}_n, \mathcal{E}_m}$$

$$\underset{1 \times 1}{\underset{\text{1x1}}{\text{1x1}}} \quad \underset{1 \times 1}{\underset{\text{1x1}}{\text{1x1}}}$$

$$\Rightarrow [T]_{\mathcal{E}_n, \mathcal{E}_m} = A$$

Why? $M_A e_j = A \begin{bmatrix} 1 \\ \vdots \\ j \\ \vdots \\ 0 \end{bmatrix} = a_j = [e_1, \dots, e_m] \begin{bmatrix} a_{1j} \\ \vdots \\ a_{mj} \end{bmatrix}$

$Ax = b \Rightarrow$ coefficients of x on e_1, \dots, e_n
 \mapsto coefficients of b on e_1, \dots, e_m

j^{th} column of $[T]_{\mathcal{V}, \mathcal{W}}$ lists the coefficients on w_1, \dots, w_m in Tv_j

Lemma: Fix V with basis v_1, \dots, v_n . If v'_1, \dots, v'_n is another basis of V

then $[v'_1, \dots, v'_n] = [v_1, \dots, v_n] P$ for an invertible $P \in \mathbb{R}^{n \times n}$. Note: $P = [\text{id}_V]_{\mathcal{V}', \mathcal{V}}$

Pf: $P = [\text{id}_V]_{\mathcal{V}', \mathcal{V}}$ by def; need P invertible. Enough: $N(P) = 0$. But

$x \in N(P) \Rightarrow [v'_1, \dots, v'_n] x = [v_1, \dots, v_n] P x = 0 \Rightarrow x = 0$ because v_1, \dots, v_n independent. \square

E.g. $V = \mathcal{P}_3$ $\mathcal{V} = [1, t, t^2, t^3]$ $\mathcal{V}' = [1, t-1, t^2-t, t^3-t^2] \Rightarrow P = ?$

$$[1 \ t-1 \ t^2-t \ t^3-t^2] = [1 \ t \ t^2 \ t^3] \begin{bmatrix} 1 & -1 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 1 \end{bmatrix} \xrightarrow{P} \text{change-of-basis matrix}$$

Theorem 4.2 (change-of-basis formula): Fix a linear map $T: V \rightarrow W$,

• ordered bases $\mathcal{V}, \mathcal{V}'$ for V with $[v_1 \dots v_n] = [v_1 \dots v_n] P$

$\mathcal{W}, \mathcal{W}'$ for W $[w_1 \dots w_m] = [w_1 \dots w_m] Q$,

• matrices $A = [T]_{\mathcal{V}, \mathcal{W}}$ and $A' = [T]_{\mathcal{V}', \mathcal{W}'}$. Then $A' = Q^{-1}AP$.

Pf: $T[v_1 \dots v_n] = [w_1 \dots w_m] A \Rightarrow T[v_1 \dots v_n] P = [w_1 \dots w_m] AP$

$T[v_1 \dots v_n] = [w_1 \dots w_m] Q^{-1}AP$ defines A' . \square

E.g. $T: \mathcal{P}_3 \rightarrow \mathbb{R}^2$ $T = ev_{0,1}$ $\mathcal{W} = \mathcal{E}_2 = (e_1, e_2) = \mathcal{W}'$.

$$f \mapsto \begin{bmatrix} f(0) \\ f(1) \end{bmatrix}$$

Calculate $A = [T]_{\mathcal{V}, \mathcal{W}}$ and $A' = [T]_{\mathcal{V}', \mathcal{W}'}$

and verify the relation between them.

$$A = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 \end{bmatrix} \quad \text{evaluate basis vectors at } 0 \text{ and } 1$$

$$A' = \begin{bmatrix} 1 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix} \quad \begin{array}{l} v'_3 \text{ and } v'_4 \in \ker T \\ \uparrow \quad \downarrow \\ 1e_1 + 1e_2 \quad -1e_1 + 0e_2 \end{array}$$

$$Q = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I_2$$

$$Q^{-1}AP = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}^{-1} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & -1 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix} = A'$$

aside: $\ker T$ more visible in A' :

$$v'_3 \text{ and } v'_4 \mapsto \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Rightarrow v'_3, v'_4 \text{ form a basis of } \ker T$$

$\text{image } T: Tv'_1 \text{ and } Tv'_2 \text{ span } \text{image } T \Rightarrow \text{they form a basis for } \text{im } T$

rank-nullity: $2+2=4$