

23. §6.1 Def: Fix $T: V \rightarrow V$. $v \in V$ is an eigenvector if $v \neq 0$ and

$Tv = \lambda v$ for some scalar λ , called an eigenvalue of T . works over any field

E.g. $A \in \mathbb{C}^{n \times n}$ with rank $A < n$ has eigenvalue 0 : $v \in N(A) \setminus \{0\}$ has $Av = 0v$.

Prop 1.1: B is a basis of eigenvectors of $T: V \rightarrow V \xrightarrow{\text{§4.3}}$ $[T]_B$ is diagonal.

$$\text{Pf: } T[v_1 \dots v_n] = [\lambda_1 v_1 \dots \lambda_n v_n] = [v_1 \dots v_n] \begin{bmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{bmatrix} \Leftrightarrow [T]_B = \begin{bmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{bmatrix}. \quad \square$$

\uparrow
 v_1, \dots, v_n eigenvectors \uparrow
 $[T]_B$ diagonal

Def: $T: V \rightarrow V$ is diagonalizable if $[T]_B$ is diagonal for some basis B of V .

An $n \times n$ matrix A is diagonalizable if μ_A is.

Lemma: A is diagonalizable $\Leftrightarrow A$ is similar to a diagonal matrix. A is similar to Λ

$$\text{Pf: } \mu_A[v_1 \dots v_n] = [v_1 \dots v_n] \begin{bmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{bmatrix} \Leftrightarrow A \begin{bmatrix} 1 & & \\ v_1 \dots v_n & \ddots & \\ 1 & & 1 \end{bmatrix} = \begin{bmatrix} 1 & & \\ v_1 \dots v_n & \ddots & \\ 1 & & 1 \end{bmatrix} \begin{bmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{bmatrix} \Leftrightarrow A = P \Lambda P^{-1}. \quad \square$$

E.g. not diagonalizable: $\begin{bmatrix} 2 & 1 \\ 0 & 2 \end{bmatrix}$ (proof in a bit)

Note: Why is $A = P \Lambda P^{-1}$ useful?

$$A^k = (P \Lambda P^{-1})(P \Lambda P^{-1}) \cdots (P \Lambda P^{-1}) = P \Lambda^k P^{-1} = P \begin{bmatrix} \lambda_1^k & & \\ & \ddots & \\ & & \lambda_n^k \end{bmatrix} P^{-1}$$

$$\text{Aside: } e^A = I + A + \frac{A^2}{2!} + \frac{A^3}{3!} + \cdots \quad (\text{and this converges}) - \text{see Math 403}$$

Lemma 1.2: λ is an eigenvalue of $T: V \rightarrow V$ or $n \times n A$

$$\Leftrightarrow \underbrace{\ker(T - \lambda I)}_{\lambda\text{-eigenspace of } T} \neq 0 \quad \underbrace{N(A - \lambda I)}_{\dots \text{ of } A} \neq 0 \quad E(\lambda)$$

$$\text{Pf: } v \neq 0 \text{ satisfies } \underbrace{T v}_{A} = \lambda v \Leftrightarrow \underbrace{T v - \lambda v}_{A} = 0 \Leftrightarrow$$

$$\Leftrightarrow \underbrace{(T - \lambda I)v}_{A} = 0. \quad \square$$

$v \in E(\lambda) \setminus \{0\} \Leftrightarrow \underbrace{T v}_{A} = \lambda v \text{ and } v \neq 0$
 $\Leftrightarrow v \text{ is an eigenvector with eigenvalue } \lambda$

Prop: λ is an eigenvalue of $A \Leftrightarrow \det(A - \lambda I) = 0$.

Pf: $N(A - \lambda I) \neq 0 \Leftrightarrow A - \lambda I$ singular

$$\Leftrightarrow \det(A - \lambda I) = 0. \quad \square$$

E.g. Eigenvalues of $A = \begin{bmatrix} 2 & 1 \\ 0 & 2 \end{bmatrix}$ are the roots of

$$\det\left(\begin{bmatrix} 2 & 1 \\ 0 & 2 \end{bmatrix} - t \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}\right) = \det\left(\begin{bmatrix} 2-t & 1 \\ 0 & 2-t \end{bmatrix}\right) = (2-t)^2$$

so λ eigenvalue $\Leftrightarrow \lambda = 2$. But $E(2) = N\left(\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}\right) \neq 0$ and $\neq \mathbb{R}^2$

$\Rightarrow \dim E(2) = 1 \Rightarrow$ no basis of eigenvectors.

E.g. Find eigenvalues and eigenvectors of $\begin{bmatrix} 3 & 1 \\ -3 & 7 \end{bmatrix}$.

$$\begin{aligned} \det(A - tI) &= \det \begin{bmatrix} 3-t & 1 \\ -3 & 7-t \end{bmatrix} = (3-t)(7-t) + 3 \\ &= 21 - 7t - 3t + t^2 + 3 = t^2 - 10t + 24 \\ &= (t-4)(t-6) \end{aligned}$$

λ eigenvalue $\Leftrightarrow \lambda = 4$ or $\lambda = 6$

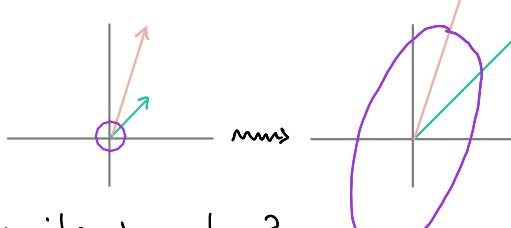
$$E(4) = N(A - 4I) = N \begin{bmatrix} -1 & 1 \\ -3 & 3 \end{bmatrix} = \text{span} \left(\begin{bmatrix} 1 \\ 1 \end{bmatrix} \right)$$

eigenvector

check $A \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

$$E(6) = N(A - 6I) = N \begin{bmatrix} -3 & 1 \\ -3 & 1 \end{bmatrix} = \text{span} \left(\begin{bmatrix} 1 \\ 3 \end{bmatrix} \right)$$

check $A \begin{bmatrix} 1 \\ 3 \end{bmatrix}$



You can "see" what μ_A does!

E.g. $A = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$. What are its eigenvalues?

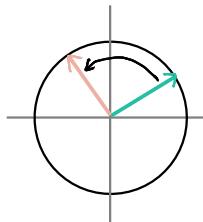
$$\begin{aligned} \det(A - tI) &= \det \begin{bmatrix} -t & -1 \\ 1 & -t \end{bmatrix} = t^2 + 1 \\ &= 0 \Leftrightarrow t^2 = -1 \end{aligned}$$

$\Leftrightarrow t = i$ or $t = -i \Rightarrow$ no real eigenvalues!

Q. Why not? What map does A represent?

A. rotation by $\pi/2$ has no real eigenvectors:

$v \neq 0$ moves to a different (orthogonal) line,
not a scalar multiple



But: $A \begin{bmatrix} 1 \\ i \end{bmatrix} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ i \end{bmatrix} = \begin{bmatrix} -i \\ 1 \end{bmatrix} = -i \begin{bmatrix} 1 \\ i \end{bmatrix}$! eigenvector with eigenvalue $-i$!

$A \begin{bmatrix} 1 \\ -i \end{bmatrix} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ -i \end{bmatrix} = \begin{bmatrix} i \\ 1 \end{bmatrix} = i \begin{bmatrix} 1 \\ -i \end{bmatrix}$ eigenvector with eigenvalue i !

"multiplication by i is rotation by $\pi/2$!"

Def: The characteristic polynomial of an $n \times n$ matrix A is $p_A(t) = \det(A - tI)$.

So root of $p_A(t) \Leftrightarrow$ eigenvalue of A .

Lemma 1.4: A similar to B $\Rightarrow p_A(t) = p_B(t)$.

Pf: Suppose $B = QAQ^{-1}$. Then $p_B(t) = p_{QAQ^{-1}}(t) = \det(QAQ^{-1} - tI)$

$$\begin{aligned} &= \det(QAQ^{-1} - Q(tI)Q^{-1}) \\ &= \det(Q(A - tI)Q^{-1}) \\ &= \cancel{\det Q} \det(A - tI) \cancel{\det Q^{-1}} \\ &= p_A(t). \square \end{aligned}$$

Def: $T: V \rightarrow V$ has characteristic polynomial $p_T = p_A$ for some (hence every) $A = [T]_B$. Well defined by Lemma 1.4.

$$p_A(t) = c_n t^n + c_{n-1} t^{n-1} + \dots + c_1 t + c_0.$$

Q. What's c_n ?

A. $(-1)^n$

Q. What's c_0 ?

A. $c_0 = p_A(0) = \det(A - 0I) = \det A$.

A' A diagonalizable, say $A \sim \Lambda$, $\Rightarrow p_A(t) = p_\Lambda(t) = \det \begin{bmatrix} \lambda_1 - t & & \\ & \ddots & \\ & & \lambda_n - t \end{bmatrix}$

$$= (\lambda_1 - t) \cdots (\lambda_n - t)$$

has $c_0 = \lambda_1 \cdots \lambda_n$.

General (next lecture): $\det A$ = product of eigenvalues with multiplicities counted correctly.

A singular $\Leftrightarrow 0$ is an eigenvalue

$$\begin{aligned} &\Leftrightarrow \text{product of eigenvalues is } 0 \\ &\Leftrightarrow \det A = 0 \end{aligned}$$

} explains why \det detects singularity