Math 221 Homework for Section 4.4: additional problems on change of basis

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1. Choose bases

   - $\mathbf{w}_0 = 1 = \mathbf{v}_0$,
   - $\mathbf{w}_1 = t = \mathbf{v}_1$,
   - $\mathbf{w}_2 = t^2 = \mathbf{v}_2$,
   - $\mathbf{w}_3 = t^3 = \mathbf{v}_3$, and
   - $\mathbf{w}_4 = t^4$

   for $V = \mathcal{P}_3$ and $W = \mathcal{P}_4$. Describe the map $T : \mathcal{P}_3 \to \mathcal{P}_4$ satisfying

   $$\begin{bmatrix} T\mathbf{v}_0 & T\mathbf{v}_1 & T\mathbf{v}_2 & T\mathbf{v}_3 \end{bmatrix} = \begin{bmatrix} \mathbf{w}_0 & \mathbf{w}_1 & \mathbf{w}_2 & \mathbf{w}_3 & \mathbf{w}_4 \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

   explicitly in terms of the algebra of polynomials.

2. In Question 1, what would the 5×4 matrix be if the basis for V were instead $(1 - t, t - t^2, t^2 - t^3, t^3)$? Answer the question by computing $T(1-t), T(t-t^2), T(t^2-t^3)$, and $T(t^3)$ directly. Place your answer in an equation whose left-hand side is the 1×4 array

   $$\begin{bmatrix} T(1 - t) & T(t - t^2) & T(t^2 - t^3) & T(t^3) \end{bmatrix}.$$

3. If $\begin{bmatrix} 1+t+t^2 & t+t^2+t^3 & t^2+t^3+t^4 & t^3+t^4 & t^4 \end{bmatrix} = \begin{bmatrix} 1 & t & t^2 & t^3 & t^4 \end{bmatrix} Q$, then what is Q?

4. Do Question 2 again using the change-of-basis formula

   $$[T]_{\mathcal{V}',\mathcal{W}'} = Q^{-1} [T]_{\mathcal{V},\mathcal{W}} P,$$

   where $\begin{bmatrix} \mathbf{v}'_1 & \cdots & \mathbf{v}'_n \end{bmatrix} = \begin{bmatrix} \mathbf{v}_1 & \cdots & \mathbf{v}_n \end{bmatrix} P$ and $\begin{bmatrix} \mathbf{w}'_1 & \cdots & \mathbf{w}'_m \end{bmatrix} = \begin{bmatrix} \mathbf{w}_1 & \cdots & \mathbf{w}_m \end{bmatrix} Q$.

5. In Question 1, what would the 5×4 matrix be if the bases for V and W were instead $(1 - t, t - t^2, t^2 - t^3, t^3)$ and $(1 + t + t^2, t + t^2 + t^3, t^2 + t^3 + t^4, t^3 + t^4, t^4)$? Place your answer in an equation whose left-hand side is the 1×4 array

   $$\begin{bmatrix} T(1 - t) & T(t - t^2) & T(t^2 - t^3) & T(t^3) \end{bmatrix}.$$

Do it how you like, but the change-of-basis formula in Question 4 is probably simplest. If your method produces a product of matrices and their inverses, there is no need to take the inverses or multiply the matrices.

6. $\mathbb{R}^{2\times 2}$ has bases

$$\mathcal{V} = \left(\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right) \text{ and } \mathcal{V}' = \left(\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \right)$$

satisfying $\begin{bmatrix} \mathbf{v}'_1 & \mathbf{v}'_2 & \mathbf{v}'_3 & \mathbf{v}'_4 \end{bmatrix} = \begin{bmatrix} \mathbf{v}_1 & \mathbf{v}_2 & \mathbf{v}_3 & \mathbf{v}_4 \end{bmatrix} P$ for some matrix P. What is P?

7. For the bases in Question 6, find $\begin{bmatrix} I \end{bmatrix}_{\mathcal{V},\mathcal{V}'}$ for the identity map $I : \mathbb{R}^{2\times 2} \to \mathbb{R}^{2\times 2}$.

8. For the bases in Question 6, find the matrix A such that

$$\mu_M \begin{bmatrix} \mathbf{v}_1 & \mathbf{v}_2 & \mathbf{v}_3 & \mathbf{v}_4 \end{bmatrix} = \begin{bmatrix} \mathbf{v}'_1 & \mathbf{v}'_2 & \mathbf{v}'_3 & \mathbf{v}'_4 \end{bmatrix} A,$$

where $\mu_M : \mathbb{R}^{2\times 2} \to \mathbb{R}^{2\times 2}$ is left multiplication by the matrix $M = \begin{bmatrix} 2 & 1 \\ -1 & 3 \end{bmatrix}$.

9. For the bases in Question 6, find $\begin{bmatrix} \mu_M \end{bmatrix}_{\mathcal{V}',\mathcal{V}}$ for the map $\mu_M : \mathbb{R}^{2\times 2} \to \mathbb{R}^{2\times 2}$ that is left multiplication by M, where $M = \begin{bmatrix} 2 & 1 \\ -1 & 3 \end{bmatrix}$. How is this problem different from Question 8? Which was easier?