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Gröbner geometry of Schubert polynomials

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Abstract

Given a permutation $w \in S_n$, we consider a determinantal ideal I_w whose generators are certain minors in the generic $n \times n$ matrix (filled with independent variables). Using 'multidegrees' as simple algebraic substitutes for torus-equivariant cohomology classes on vector spaces, our main theorems describe, for each ideal I_w:

- variously graded multidegrees and Hilbert series in terms of ordinary and double Schubert and Grothendieck polynomials;
- a Gröbner basis consisting of minors in the generic $n \times n$ matrix;
- the Stanley–Reisner simplicial complex of the initial ideal in terms of known combinatorial diagrams [FK96], [BB93] associated to permutations in S_n; and
- a procedure inductive on weak Bruhat order for listing the facets of this complex.

We show that the initial ideal is Cohen–Macaulay, by identifying the Stanley–Reisner complex as a special kind of "subword complex in S_n", which we define generally for arbitrary Coxeter groups, and prove to be shellable by giving an explicit vertex decomposition. We also prove geometrically a general positivity statement for multidegrees of subschemes.

Our main theorems provide a geometric explanation for the naturality of Schubert polynomials and their associated combinatorics. More precisely, we apply these theorems to:

- define a single geometric setting in which polynomial representatives for Schubert classes in the integral cohomology ring of the flag manifold are determined uniquely, and have positive coefficients for geometric reasons;

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- rederive from a topological perspective Fulton's Schubert polynomial formula for universal cohomology classes of degeneracy loci of maps between flagged vector bundles;

- supply new proofs that Schubert and Grothendieck polynomials represent cohomology and K-theory classes on the flag manifold; and

- provide determinantal formulae for the multidegrees of ladder determinantal rings.

The proofs of the main theorems introduce the technique of "Bruhat induction", consisting of a collection of geometric, algebraic, and combinatorial tools, based on divided and isobaric divided differences, that allow one to prove statements about determinantal ideals by induction on weak Bruhat order.

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