The Alexander duality functors and local duality with monomial support

Alexander duality is made into a functor which extends the notion for monomial ideals to any finitely generated $\mathbb{N}^n$-graded module. The functors associated with Alexander duality provide a duality on the level of free and injective resolutions, and numerous Bass and Betti number relations result as corollaries. A minimal injective resolution of a module M is equivalent to the injective resolution of its Alexander dual, and contains all of the maps in the minimal free resolution of M over every $\mathbb{Z}^n$-graded localization. Results are obtained on the interaction of duality for resolutions with cellular resolutions and lcm-lattices. Using injective resolutions, theorems of Eagon-Reiner and Terai are generalized to all $\mathbb{N}^n$-graded modules: the projective dimension of M equals the *support-regularity* of its Alexander dual, and M is Cohen-Macaulay if and only if its Alexander dual has a *support-linear* free resolution. Alexander duality is applied in the context of the $\mathbb{Z}^n$-graded local cohomology functors $H^i_I(-)$ for squarefree monomial ideals I in the polynomial ring S, proving a duality directly generalizing local duality, which is the case when $I = \mathfrak{m}$ is maximal. In the process, a new flat complex for calculating local cohomology at monomial ideals is introduced, showing, as a consequence, that Terai's formula for the Hilbert series of $H^i_I(S)$ is equivalent to Hochster's for $H^{n-i}_{\mathfrak{m}}(S/I)$.