Calc 2 Self-Assessment

<u>Instructions:</u> This self-assessment is designed for incoming Duke students who have credit on their transcript for Math 22 or Math 122 (Calculus II), but are unsure about whether they should use this credit. For the most accurate guidance, we recommend that you

- Work through the questions below in a single 2-hour sitting;
- Write your solutions by hand on a separate sheet of paper, showing all work so that your reasoning and process are clear;
- Do not make use of any outside resources, including textbooks, calculators, the internet, and other people.

After the 2-hour period ends, click on the link to the answers, and grade your work.

1. Evaluate the following limits:

   (a) $\displaystyle\lim_{x\to 0} \frac{e^x - e^{-x} - 2x}{x - \sin(x)}$

   (b) $\displaystyle\lim_{x\to 1} \frac{x^2 + 3^{-x}}{3x + \ln(x)}$

   (c) $\displaystyle\lim_{x\to 1} \ln(x)\sec\left(\frac{\pi x}{2}\right)$

   (d) $\displaystyle\lim_{x\to 0} (\csc(x) - \cot(x))$

   (e) $\displaystyle\lim_{x\to \infty} \frac{5e^{-x} + e^{3x}}{8e^{3x} + e^{-x}}$

   (f) $\displaystyle\lim_{n\to \infty} \sum_{k=1}^{n} \left[\frac{1}{1 + (k/n)^2}\right]\frac{1}{n}$

2. Suppose $f(1) = 2$, $f'(1) = 3$, $g(1) = 1$, $g'(1) = 4$, and $g''(1) = 5$.

   (a) Find the derivative of the following functions at $x = 1$:

      i. $\sqrt{f(x)}$

      ii. $f\left(\sqrt{x}\right)$

      iii. $e^{f(x)g(x)}$

      iv. $2^{f(x)}g(x)$

      v. $\displaystyle\frac{g(x)}{g'(x)}$

   (b) Evaluate the following limits:

      i. $\displaystyle\lim_{x\to 1} \frac{f(x) - f(1)}{x - 1}$

      ii. $\displaystyle\lim_{h\to 0} \frac{g''(1)g'(1+h) - g''(1)g'(1) + hg(1+h)}{h}$

      iii. $\displaystyle\lim_{h\to 0} \frac{2^{g(1+h)} - 2^{g(1)}}{h}$

3. Suppose $f(x)$ is an **odd**, differentiable function such that we know the following:

$$f(0) = 1 \qquad f(2) = 5 \qquad f(4) = -8 \qquad f'(0) = -2 \qquad f'(2) = -1$$

$$\int_0^1 f(x)\ dx = 4 \qquad \int_0^2 f(x)\ dx = 6 \qquad \int_1^4 f(x)\ dx = 10$$

   (a) Evaluate the following:

      i. $\displaystyle\int_2^4 f(x)\ dx$

      ii. $\displaystyle\int_{-1}^1 (f(x) + 3)\ dx$

      iii. $\displaystyle\int_0^2 (f'(x)f'(x) + f(x)f''(x))\ dx$

   (b) Find $\displaystyle\frac{d}{dx}\int_0^{\sqrt{x}} f(t)\ dt$ at $x = 4$

   (c) Find $\displaystyle\frac{d}{dx}\int_0^2 f(t)\ dt$

   (d) Find $\displaystyle\int_0^2 \frac{d}{dt} f(t)\ dt$

4. For each of the following, evaluate the integral or show that it diverges:

   (a) $\displaystyle\int_1^\infty \frac{1}{(x+1)(2x+3)}\ dx$

   (b) $\displaystyle\int_2^4 \frac{1}{(x-5)(x-3)}\ dx$

5. Find the sum of the following series, simplifying your answer as much as possible.

   (a) $\displaystyle\left(\frac{1}{e} - 1\right) - \frac{\left(\frac{1}{e} - 1\right)^2}{2} + \frac{\left(\frac{1}{e} - 1\right)^3}{3} - \frac{\left(\frac{1}{e} - 1\right)^4}{4} + \frac{\left(\frac{1}{e} - 1\right)^5}{5} - \cdots$

   (b) $\displaystyle\frac{1}{2} - \frac{1}{4} + \frac{1}{8} - \frac{1}{16} + \cdots$

   (c) $\displaystyle\ln(2) + (\ln(2))^2 + \frac{(\ln(2))^3}{2!} + \frac{(\ln(2))^4}{3!} + \frac{(\ln(2))^5}{4!} + \cdots$

   (d) $\displaystyle\pi + \frac{\sqrt{\pi^3}}{2!} + \frac{\sqrt{\pi^4}}{3!} + \frac{\sqrt{\pi^5}}{4!} + \cdots$

   (e) $\displaystyle 1 + 2 - \frac{(3)^2}{2!} - \frac{(2)^3}{3!} + \frac{(3)^4}{4!} + \frac{(2)^5}{5!} + \cdots$

6. Let $\displaystyle\sum_{k=1}^{\infty} a_k$ be a series with nth partial sum $S_n = 1 - \dfrac{n}{2n+4}$. Find each of the following. Make sure to fully justify your answers.

(a) a_1

(b) $\displaystyle\sum_{k=101}^{200} a_k$

(c) $\displaystyle\sum_{k=1}^{\infty} a_k$

(d) $\displaystyle\lim_{k\to\infty} a_k$

7. Suppose $\displaystyle\int_0^1 f(x)\,dx = 3$, $f(0) = 10$, $f(1) = 4$, $f'(0) = 1$, $f'(1) = -2$, $f''(0) = 5$, $f''(1) = 8$. Evaluate the following:

(a) $\displaystyle\int_0^1 xf''(x)\,dx$

(b) $\displaystyle\int_0^1 f''(x)\Big(f'(x)\Big)^2\,dx$

(c) $\displaystyle\int_0^1 \frac{f''(x)}{f'(x)}\,dx$

(d) $\displaystyle\int_0^1 \frac{f(x)f''(x) - f'(x)f'(x)}{\Big(f(x)\Big)^2}\,dx$

8. Find the values of a and of b for which $y = ax^2 - bx$ is a solution to the initial value problem

$$x\frac{dy}{dx} - 3x - 2y = 0, \qquad y(1) = 1$$

9. Suppose $f'(x) = \dfrac{\Big(f(x)\Big)^2}{x}$, $\quad f(1) = 2$

(a) Find the 2nd degree Taylor polynomial for $f(x)$, centered at $x = 1$.

(b) Find the solution to this initial value problem.

10. Consider the series $\displaystyle\sum_{k=1}^{\infty} \frac{(-1)^k}{k^p}$. For which values of p does this series:

(a) converge absolutely?

(b) converge conditionally?

(c) diverge?