

Instructions: This self-assessment is designed for incoming Duke students who have credit on their transcript for Math 21 or Math 111 (Calculus I), but are unsure whether they should use this credit. For the most accurate guidance, we recommend that you

- Work through the 9 questions below within a single 2-hour sitting;
- Write your solutions by hand on a separate sheet of paper, showing all work so that your reasoning and process are clear;
- Do not make use of any outside resources, including textbooks, calculators, the internet, and other people.

When finished or the 2 hours end, click on the link to the answers and score your work.

1. Evaluate the following 5 limits.

(a)  $\lim_{x \rightarrow 0} \frac{e^x - e^{-x} - 2x}{x - \sin x}$

(b)  $\lim_{x \rightarrow 1} \frac{x^2 + 3^{-x}}{3x + \ln(x)}$

(c)  $\lim_{x \rightarrow 0} (\csc x - \cot x)$

(d)  $\lim_{x \rightarrow \infty} \frac{5e^{-x} + e^{3x}}{8e^{3x} + e^{-x}}$

(e)  $\lim_{n \rightarrow \infty} \sum_{k=1}^n \left( \frac{1}{1 + \left(\frac{k}{n}\right)^2} \right) \frac{1}{n}$

2. Consider the following function, where  $a$  and  $b$  are constants:

$$f(x) = \begin{cases} 2a^x & \text{if } x < 0 \\ bx + b & \text{if } x \geq 0 \end{cases}$$

Find values of  $a$  and  $b$  such that  $f(x)$  is differentiable.

3. Consider the function  $f(x) = \frac{5x^2}{1 - x^2}$ .

- (a) Find each horizontal asymptote, if any, of the graph  $f(x)$ .
- (b) Find each vertical asymptote, if any, of the graph  $f(x)$ .
- (c) Find the holes, if any, on the graph of  $f(x)$ .
- (d) Find the critical points, if any, of  $f(x)$  and classify each as a local minimum, local maximum, or neither.

4. Suppose that  $f$  and  $g$  are invertible, differentiable functions with

$$f(1) = 2, \quad f'(1) = 3, \quad f''(1) = -2, \quad g(1) = 1, \quad g'(1) = 4, \quad g''(1) = 5.$$

- (a) Find the derivative of the following 4 functions at  $x = 1$ :

$$\text{i. } \ln(f(x)) \quad \text{ii. } \sqrt{f(x)}g(\sqrt{x}) \quad \text{iii. } \frac{g(x)}{g'(x)} \quad \text{iv. } g^{-1}(x)$$

- (b) Evaluate the following 2 limits.

$$\text{i. } \lim_{x \rightarrow 1} \frac{f(x) - f(1)}{x - 1} \quad \text{ii. } \lim_{h \rightarrow 0} \frac{2^{g(1+h)} - 2^{g(1)}}{h}$$

- (c) Use linear approximation to estimate  $f(1.1)$ .

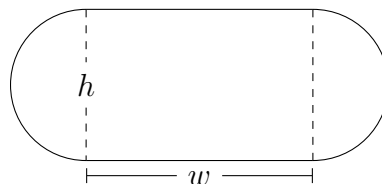
- (d) Is your estimation in part (c) an overestimate or an underestimate?

5. The equation  $4x^2 - 4xy + 4y^2 = 3$  defines an ellipse.

- (a) Find  $\frac{dy}{dx}$ .

- (b) Find each point, if any, on the ellipse at which the tangent line is vertical.

6. A playing field is to be constructed in the shape of a rectangle with a semicircle on each end, as pictured at right. What dimensions  $h$  and  $w$  give a maximum area if the perimeter must be 400 meters? Be sure to justify that your dimensions yield a global maximum.



7. A 10-meter tall ladder leans against a vertical wall and the bottom of the ladder slides away from the wall at a rate of 0.5 m/sec. How fast is the top of the ladder sliding down the wall when the angle between the ladder and the floor is  $45^\circ$ ?
8. Consider the functions  $f(t) = t \cos(2t)$ ,  $g(x) = 2x^2 + 1$ , and  $h(x) = e^{2x}$ .

- (a) Suppose  $F(x) = \int_2^{x^2} f(t)dt$ . Find  $F'(3)$ .

- (b) Find the average value of  $g(x)$  on the interval  $[0, 2]$ .

- (c) Find the area of the region bounded by the graphs of  $g(x)$ ,  $h(x)$ , and  $x = 1$ .

9. The rate of metabolization and excretion (i.e., removal) of caffeine from a person's body is proportional to the current amount of caffeine in the body, with constant of proportionality  $k$ . Suppose a studying student with no caffeine in her body starts drinking coffee, containing 120 mg of caffeine per cup, at 6am. The student then continues drinking coffee continuously at a rate of 0.5 cups per hour.
- (a) Write an initial value problem (a differential equation plus an initial condition) satisfied by  $F(t)$ , the amount of caffeine, in mg, in the student's body  $t$  hours after 6am.
- (b) Suppose that at the instant in time the amount of caffeine in the student's body is 100mg, the amount is increasing at a rate of 40 mg/hr. Find the value of the constant of proportionality  $k$ .
- (c) Solve your initial value problem to find a model for  $F(t)$ .