Bad Maths That Give The Right Answer

2020 DMM Power Round

Introduction

Consider the following scenario: A student is asked to simplify the fraction $\frac{16}{64}$, and computes

$$\frac{1\not{6}}{\not{6}4} = \frac{1}{4}.$$

As we all know, the method is incorrect, but the answer is correct. In this round, we will explore these "lucky fractions" in greater detail.

Lucky Triples

Define a *lucky triple* (x, y, z) to be a triple such that

$$\frac{\overline{xy}}{\overline{yz}} = \frac{x}{z},$$

where $\overline{xy}$ denotes the two-digit number xy in base 10. In this way, if a student incorrectly "cancels out the y's", they would be lucky and still get the correct answer. As shown above, $(1, 6, 4)$ is an example of a lucky triple. In order to rule out trivial lucky triples, we require that $x, y, z \geq 1$, and we don't count triples where $x = y = z$. So, although $\frac{22}{22} = \frac{2}{2}$ and $\frac{00}{04} = \frac{0}{4}$, both $(2, 2, 2)$ and $(0, 0, 4)$ are *not* lucky triples.

[3] **Problem 1.** Find the other three lucky triples.

This quickly gets rather boring, so we make the problem more interesting by considering lucky triples in different bases. Let us define a *lucky b-triple* to be a triple in base b such that

$$\frac{\overline{xy}}{\overline{yz}} = \frac{x}{z}.$$

In other words, we want (x, y, z) to be integers that satisfy

$$\frac{b \cdot x + y}{b \cdot y + z} = \frac{x}{z}.$$

Again, we have the restrictions that $1 \leq x, y, z < b$, and we do not count the case of $x = y = z$. We also define the function $\lambda(b)$ to be the number of lucky b-triples. For example, $\lambda(10) = 4$, and the four lucky triples are $(1, 6, 4)$, and the three that you computed in problem 1. For the remainder of this section, we will mostly practice computing $\lambda(b)$ and finding lucky b-triples for small values of b to gain some intuition about this strange function.

Problem 2. Find, with proof, the following values:

[2] (a) $\lambda(3)$

[3] (b) $\lambda(4)$

[3] (c) $\lambda(6)$

[4] (d) $\lambda(9)$

[4] **Problem 3.** Find, with proof, the smallest base b such that $\lambda(b) > 2$.

[3] **Problem 4.** Find a triple (x, y, z) such that (x, y, z) is a lucky b_1-triple and a lucky b_2-triple, where $b_1 \neq b_2$, or prove that none exist.

Properties of λ

Now that we have some practice with computing the values of the function λ, let us explore some properties about the λ function. After all, calculations are worthless if we learn nothing from them.

[2] **Problem 5.** Prove that if p is prime, then $\lambda(p) = 0$.

Problem 6. Recall that a proper factor of b is a factor other than 1 or b. For example, 2 and 3 are proper factors of 6, but 1 and 6 are not.

[2] (a) Prove that if (x, y, z) is a lucky b-triple, where $b - 1$ is prime, then $y = b - 1$.

[2] (b) Prove that when $b - 1$ is prime, $\lambda(b)$ is exactly equal to the number of proper factors of b.

[1] (c) Prove that $\lambda(b)$ is greater than or equal to the number of proper factors of b.

[2] (d) Prove that if b is odd and not prime, $\lambda(b)$ is greater than the number of proper factors of b.

Problem 7. With the lucky 10-triple $(x, y, z) = (1, 6, 4)$, note that we have $2x \leq z \leq y$.

[4] (a) Prove that if (x, y, z) is a lucky b-triple, then the above inequality must hold, that is, $2x \leq z \leq y$.

[2] (b) Can any of the inequalities above be strict? In other words, can we prove that $2x < z$ or $z < y$ for all lucky b-triples (x, y, z)?

An Algorithmic Approach

The earlier results give us insights into how to generate lucky triples with $y = b - 1$. In this section, we will develop an algorithm to find all the non-trivial lucky b-triples with $y < b - 1$. Assume that (x, y, z) is a lucky b-triple. We will start by picking a prime factor p of $b - 1$, and we define $l = \frac{b-1}{p}$.

[1] **Problem 8.** Prove that either $p \mid y$ or $p \mid bx - z$.

[3] **Problem 9.** Prove that if $p \mid bx - z$, then there exists a prime q such that q is a factor of $b - 1$ and y is divisible by q.

Problem 10. The result of problem 9 suggest that we only need to consider the case of $p \mid y$. Given this assumption, prove the following:

[4] (a) Let us define $m = \frac{y}{p}$ and $k = \frac{yz}{px} = \frac{mz}{x}$. Find a lucky b-triple in terms of p, b, m, k, and l.

[2] (b) Prove that $k \equiv m \mod l$, that is, m and k have the same remainder when divided by l.

[3] (c) Find the best upper and lower bounds for k in terms of b, m, and l. Note that while 0 is obviously a lower bound for k, it is not the best lower bound.

Using the above results, we can delineate a quick algorithm to find all lucky b-triples with $y < b - 1$. As an interesting result, we can prove that any lucky b-triple (x, y, z) must satisfy $\gcd(y, b - 1) > 1$.

Final thoughts

The following problems are considered extra credit, and not answering them will not negatively impact your score. However, due to the extreme difficulty of the problems, we *strongly* recommend finishing the earlier problems before tackling these for the sake of using your time efficiently.

Problem 11. Prove or disprove the following:

[5] (a) $\lambda(b)$ is odd if and only if $b = 4n^2$ for some positive integer n.

[5] (b) There exists infinite values of n such that there does not exist a b with $\lambda(b) = n$.

[5] (c) $\lambda(b) < b$ for all b.

[5] (d) There are infinitely many b such that $\lambda(b) = 2$.