Bad Maths That Give The Right Answer

2020 DMM Power Round

Introduction

Consider the following scenario: A student is asked to simplify the fraction $\frac{16}{64}$, and computes

$$\frac{1\not{6}}{\not{6}4} = \frac{1}{4}.$$

As we all know, the method is incorrect, but the answer is correct. In this round, we will explore these "lucky fractions" in greater detail.

Lucky Triples

Define a *lucky triple* (x, y, z) to be a triple such that

$$\frac{\overline{xy}}{\overline{yz}} = \frac{x}{z},$$

where $\overline{xy}$ denotes the two-digit number xy in base 10. In this way, if a student incorrectly "cancels out the y's", they would be lucky and still get the correct answer. As shown above, $(1, 6, 4)$ is an example of a lucky triple. In order to rule out trivial lucky triples, we require that $x, y, z \geq 1$, and we don't count triples where $x = y = z$. So, although $\frac{22}{22} = \frac{2}{2}$ and $\frac{00}{04} = \frac{0}{4}$, both $(2, 2, 2)$ and $(0, 0, 4)$ are *not* lucky triples.

[3] **Problem 1.** Find the other three lucky triples.

> **Solution:** The other three lucky triples are (1, 9, 5), (2, 6, 5), and (4, 9, 8).

This quickly gets rather boring, so we make the problem more interesting by considering lucky triples in different bases. Let us define a *lucky b-triple* to be a triple in base b such that

$$\frac{\overline{xy}}{\overline{yz}} = \frac{x}{z}.$$

In other words, we want (x, y, z) to be integers that satisfy

$$\frac{b \cdot x + y}{b \cdot y + z} = \frac{x}{z}.$$

Again, we have the restrictions that $1 \leq x, y, z < b$, and we do not count the case of $x = y = z$. We also define the function $\lambda(b)$ to be the number of lucky b-triples. For example, $\lambda(10) = 4$, and the four lucky triples are $(1, 6, 4)$, and the three that you computed in problem 1. For the remainder of this section, we will mostly practice computing $\lambda(b)$ and finding lucky b-triples for small values of b to gain some intuition about this strange function.

Problem 2. Find, with proof, the following values:

[2] (a) $\lambda(3)$

> **Solution:** $\lambda(3) = 0$. The only possible triples are $(1, 1, 2), (1, 2, 1), (2, 1, 1), (1, 2, 2), (2, 1, 2)$, and $(1, 2, 2)$, none of which are satisfy the condition.

[3] (b) $\lambda(4)$

Solution: $\lambda(4) = 1$. The condition is equivalent to

$$4x(z - y) = z(x - y).$$

Note that since $|x - y| \leq 2$, and $1 \leq z \leq 3$, we have $|z(x - y)| \leq 6$, so $|z(x - y)|$ must be equal to 4 or 0. But, $|z(x - y)| = 0$ implies $x = y$ and $y = z$, which is impossible. Then, we get $z = |x - y| = 2$, and $x = |z - y| = 1$. Then, we have $z = 2, x = 1, y = 3$, giving us $(1, 3, 2)$ as the only lucky 4-triple.

[3] (c) $\lambda(6)$

Solution: $\lambda(6) = 2$. Again, we have

$$6x(z - y) = z(x - y),$$

and $|z(x - y)| \leq 5 \cdot 4 = 20$, so $|z(x - y)| = 6$ or 12. Note that $|z(x - y)| \neq 18$ because $z, |x - y| < 6$. If $|z(x - y)| = 6$, then $|x(z - y)| = 1$, so $x = 1$. Furthermore, either $z = 3$ or $z = 2$, neither of which gives us a valid value for y. If $|z(x - y)| = 12$, we have $z = 3$ or $z = 4$. Furthermore, $|x(z - y)| = 2$, so either $x = 1$ or $x = 2$. Then, we find that the only lucky 6-triples are $(2, 5, 4)$ and $(1, 5, 3)$.

[4] (d) $\lambda(9)$

Solution: $\lambda(9) = 2$. We have

$$9x(z - y) = z(x - y),$$

and $|z(x-y)| \leq 56$. Since 9 must divide $|z(x-y)|$, and $z \leq 8$ and $|x-y| \leq 7$, we have $|z(x-y)| = 9, 18,$ or 36, giving us $|x(z - y)| = 1, 2,$ or 4, respectively. Notice that if $|z(x - y)| = 9, 18,$ or 36, then z can only be 3 or 6, further reducing the triples we need to check. Then, we simply check all triples that satisfy these conditions and find that $(1, 4, 3)$ and $(2, 8, 6)$ are the only lucky 9-triples.

[4] **Problem 3.** Find, with proof, the smallest base b such that $\lambda(b) > 2$.

Solution: $b = 10$ is the smallest b such that $\lambda(b) > 2$. For $b = 5$ and $b = 7$, we have $bx(z - y) = z(x - y)$, where $z < b$ and $x - y < b - 1$. However, since both 5 and 7 are prime, we must have $b \mid z$ or $b \mid x - y$, which is not possible unless $x - y = 0$. In this case, we would have $z - y = 0$, so $x = y = z$, a contradiction. Thus, both $\lambda(5) = 0$ and $\lambda(7) = 0$.

We now only need to check $\lambda(8)$ because we know $\lambda(10) = 4$. The condition for a lucky 8-triple is then equivalent to

$$(8x - z)y = 7xz.$$

But, $8x - z \equiv x - z \equiv 0 \mod 7$ only when $x = z$, but there are no non-trivial lucky triples with $x = z$. Therefore, since $0 < y \leq 7$ and $7|y$, we must have $y = 7$. The equation thus simplifies to $8x - z = xz$, or $(x + 1)(z - 8) = -8$. We can then easily find that the only lucky 8-triples are $(3, 7, 6)$ and $(1, 7, 4)$, so $\lambda(8) = 2$.

[3] **Problem 4.** Find a triple (x, y, z) such that (x, y, z) is a lucky b_1-triple and a lucky b_2-triple, where $b_1 \neq b_2$, or prove that none exist.

Solution: Suppose for the sake of contradiction that there exists such a triple (x, y, z). Then, we have

$$\frac{b_1x + y}{b_1y + z} = \frac{x}{z} = \frac{b_2x + y}{b_2y + z},$$

which is equivalent to

$$b_1xz + b_2y^2 = b_2xz + b_1y^2 \implies (b_2 - b_1)xz = (b_2 - b_1)y^2 \implies y^2 = xz.$$

Then, we know that

$$y(b_1x - z) = xz(b_1 - 1) \implies b_1x - z = y(b_1 - 1).$$

Since $b_1 - 1$ divides $b_1x - z$ iff $b_1 - 1$ divides $x - z$ iff $x = z$, we have

$$b_1x - z = y(b_1 - 1) \implies x(b_1 - 1) = y(b_1 - 1) \implies x = y,$$

which is a contradiction. Therefore, there are no triples (x, y, z) such that (x, y, z) is both a lucky b_1-triple and a lucky b_2-triple, where $b_1 \neq b_2$.

Properties of λ

Now that we have some practice with computing the values of the function λ, let us explore some properties about the λ function. After all, calculations are worthless if we learn nothing from them.

[2] **Problem 5.** Prove that if p is prime, then $\lambda(p) = 0$.

Solution: The condition is equivalent to

$$px(z - y) = z(x - y).$$

If p is prime, then p must divide either z or $x - y$, but $z \leq p - 1$ and $|x - y| \leq p - 2$, so the only way for (x, y, z) to be a lucky triple is if $x - y = 0$, or $x = y$. But, that would require $px(z - y) = 0$, or $z = y$. This is a contradiction, so there are no lucky p-triples if p is prime.

Problem 6. Recall that a proper factor of b is a factor other than 1 or b. For example, 2 and 3 are proper factors of 6, but 1 and 6 are not.

[2] (a) Prove that if (x, y, z) is a lucky b-triple, where $b - 1$ is prime, then $y = b - 1$.

Solution: If (x, y, z) is a lucky b-triple, then

$$(bx - z)y = (b - 1)xz,$$

and if $b - 1$ is prime, then $b - 1 \mid y$ or $b - 1 \mid bx - z$. However, $bx - z \equiv x - z \mod b - 1$, which is only congruent to 0 if $x = z$, which is impossible. Thus, we must have $b - 1 \mid y$, and since $1 \leq y \leq b - 1$, the only value of y that works is $y = b - 1$.

[2] (b) Prove that when $b - 1$ is prime, $\lambda(b)$ is exactly equal to the number of proper factors of b.

Solution: By the result in part (a), we have to find x and z that satisfy $bx - z = xz$, so $(x + 1)(b - z) = b$. Then, $b - z \mid b$, and since $1 \leq z < b - 1$ ($z \neq b - 1$ because otherwise $x = z = b - 1$, a contradiction), $b - z$ is a proper factor of b, so there is exactly one lucky b-triple for every proper factor of b, from which the desired statement follows.

[1] (c) Prove that $\lambda(b)$ is greater than or equal to the number of proper factors of b.

> **Solution:** From the previous proof, we can see that for every proper factor k of b, $(\frac{b}{k}-1, b-1, b-k)$ is a lucky b-triple whenever $b-1$ is prime. These triples also work for any b, so $\lambda(b)$ must be at least the number of proper factors of b.

[2] (d) Prove that if b is odd and not prime, $\lambda(b)$ is greater than the number of proper factors of b.

> **Solution:** We know now that $(\frac{b}{k}-1, b-1, b-k)$ is a lucky triple for any proper factor k of b. Since b is odd, k and $\frac{b}{k}$ are both odd, so each term in the triple $(\frac{b}{k}-1, b-1, b-k)$ is even. Then, if we divide each term by 2, we get another lucky b-triple in addition to the ones where $y=b-1$. Since the number of lucky b-triples with $y=b-1$ is equal to the number of proper factors of b, $\lambda(b)$ must be greater than the number of proper factors of b.

Problem 7. With the lucky 10-triple $(x,y,z)=(1,6,4)$, note that we have $2x \le z \le y$.

[4] (a) Prove that if (x,y,z) is a lucky b-triple, then the above inequality must hold, that is, $2x \le z \le y$.

> **Solution:** We have
>
> $$bx(z-y) = z(x-y) \text{ and } (bx-z)y = (b-1)xz,$$
>
> so if $y<z$, we must have $y<x$ as well, otherwise the first equation would not hold. But, if $y<x$ and $y<z$, we have
>
> $$(bx-z)y = bxy - zy < bxz - zx = (b-1)xz,$$
>
> a contradiction with the second equation. Thus, we must have $y \ge z$. In fact, since $y=z$ implies $x=y$, which is not allowed, we must have $z<y$.
>
> Now, assume $2x>z$. Then, we have
>
> $$(b-1)xz = (bx-z)y > (b-2)xy,$$
>
> or
>
> $$\frac{b-1}{b-2}z > y.$$
>
> Combined with $y>z$, this means that there must exist an integer between z and $\frac{b-1}{b-2}z$, exclusive. In other words,
>
> $$\frac{b-1}{b-2}z - z > 1,$$
>
> so $z>b-2$, thus we must have $z=b-1$. But, this would result in $z \ge y$, a contradiction. Therefore, we must have $z \ge 2x$. Putting everything together, we have $2x \le z < y$.

[2] (b) Can any of the inequalities above be strict? In other words, can we prove that $2x<z$ or $z<y$ for all lucky b-triples (x,y,z)?

> **Solution:** We proved in part (a) that we must have $z<y$. However, we cannot prove $2x<z$. Indeed, the lucky 4-triple $(1,3,2)$ is an example where $2x=z=2$, so we cannot improve the inequality.

An Algorithmic Approach

The earlier results give us insights into how to generate lucky triples with $y = b - 1$. In this section, we will develop an algorithm to find all the non-trivial lucky b-triples with $y < b - 1$. Assume that (x, y, z) is a lucky b-triple. We will start by picking a prime factor p of $b - 1$, and we define $l = \frac{b-1}{p}$.

[1] **Problem 8.** Prove that either $p \mid y$ or $p \mid bx - z$.

Solution: We have

$$(bx - z)y = (b - 1)xz,$$

so if $p \mid b - 1$, we must have either $p \mid y$ or $p \mid bx - z$.

[3] **Problem 9.** Prove that if $p \mid bx - z$, then there exists a prime q such that q is a factor of $b - 1$ and y is divisible by q.

Solution: Since $bx - z = (b - 1)x + x - z$, and $p \mid b - 1$, it follows that $p \mid x - z$. Let us define

$$s = \frac{x - z}{p},$$

and recall that $l = \frac{b-1}{p}$. Then, $bx - z = p(lx + s)$, and since $(bx - z)y = (b - 1)xz$, we have

$$z = \frac{bx - z}{(b - 1)x}y = \frac{p(lx + s)}{plx}y = \left(1 + \frac{s}{lx}\right)y.$$

Now, $x \geq 1$, $|x - z| < b - 1$, so $|s| < l$. Thus, $\frac{sy}{lx}$ can only be an integer if $lx \mid y$. Since l is a proper factor of $b - 1$, y must be divisible by some prime factor of $b - 1$. Take this prime factor to be q, and we are done.

Problem 10. The result of problem 9 suggest that we only need to consider the case of $p \mid y$. Given this assumption, prove the following:

[4] (a) Let us define $m = \frac{y}{p}$ and $k = \frac{yz}{px} = \frac{mz}{x}$. Find a lucky b-triple in terms of p, b, m, k, and l.

Solution: Dividing both sides of $(bx - z)y = (b - 1)xz$ by p, we get

$$(bx - z)m = xzl.$$

Hence, $x \mid mz$, so $k = \frac{mz}{x}$ is an integer, and $mz = xk$. Making this substitution, we have

$$bxm - xk = xzl \implies bm = k + zl = k + \frac{lxk}{m} = k\left(1 + \frac{lx}{m}\right).$$

Consequently, $k \mid m^2 b$. Then, solving for x, we get

$$x = \frac{m^2 b - mk}{kl},$$

so

$$z = \frac{kx}{m} = \frac{mb - k}{l}.$$

This gives us the lucky b-triple

$$\left(\frac{m^2 b - mk}{kl}, mp, \frac{mb - k}{l}\right).$$

[2] (b) Prove that $k \equiv m \mod l$, that is, m and k have the same remainder when divided by l.

Solution: From the solution to part (a), we have

$$z = \frac{mb - k}{l},$$

which we will rewrite as

$$z = \frac{m(b-1) + m - k}{l}.$$

Since z is an integer and $l \mid b - 1$, we have $l \mid m - k$, so $k \equiv m \mod l$.

[3] (c) Find the best upper and lower bounds for k in terms of b, m, and l. Note that while 0 is obviously a lower bound for k, it is not the best lower bound.

Solution: Since $x \geq 1$, and from the solution to (a), we found

$$x = \frac{m^2 b - mk}{kl},$$

we have

$$\frac{m^2 b - mk}{kl} \geq 1 \implies k \leq \frac{m^2 b}{l + m}.$$

From part (b), we have $k \equiv m \mod l$, so $k \geq m$. However, if $k = m$, we have:

$$\begin{aligned} x &= \frac{m^2 b - mk}{kl} = \frac{mb - m}{l} = m\frac{b-1}{l} = mp, \\ y &= mp, \\ z &= \frac{mb - k}{l} = \frac{mb - m}{l} = mp, \end{aligned}$$

which is a trivial solution and doesn't count. Consequently, $k \geq m + l$.

Using the above results, we can delineate a quick algorithm to find all lucky b-triples with $y < b - 1$. As an interesting result, we can prove that any lucky b-triple (x, y, z) must satisfy $\gcd(y, b-1) > 1$.

Final thoughts

The following problems are considered extra credit, and not answering them will not negatively impact your score. However, due to the extreme difficulty of the problems, we *strongly* recommend finishing the earlier problems before tackling these for the sake of using your time efficiently.

Problem 11. Prove or disprove the following:

[5] (a) $\lambda(b)$ is odd if and only if $b = 4n^2$ for some positive integer n.

[5] (b) There exists infinite values of n such that there does not exist a b with $\lambda(b) = n$.

[5] (c) $\lambda(b) < b$ for all b.

[5] (d) There are infinitely many b such that $\lambda(b) = 2$.

Remark: Problem 11 currently does not have solutions, and (d) is an unsolved problem in mathematics. More specifically, if $\lambda(b) = 2$, then b has 1 or 2 proper factors. If b has 1 proper factor, then $b = p^2$, for which $(p-1, p^2-1, p^2-p)$ is a solution. But, since $p-1$ divides each term in that triple, there exists a triple for every factor of $p-1$, so the only p for which $\lambda(p^2) = 2$ is $p = 3$. If b has 2 proper factors, then b must be even, for if b is odd, then by 6 (d), $\lambda(b) > 2$. Then, $b = 2p$ for some prime p, or $b = 2^3 = 8$. If $b = 2p$ and $\lambda(b) = 2$, then we must have $2p - 1$ is also prime. Proving that there are infinite primes p such that $2p - 1$ is also prime is very closely related to the Sophie Germain primes conjecture (infinite p with $2p + 1$ prime), which is an unsolved hypothesis.