2020 Duke Math Meet

Problems and Solutions

Saturday 31^{st} October, 2020

1 Individual Problems

Problem 1.1. Four witches are riding their brooms around a circle with circumference 10m. They are standing at the same spot, and then they all start to ride clockwise with the speed of $1, 2, 3$, and 4 m/s, respectively. Assume that they stop at the time when every pair of witches has met for at least two times (the first position before they start counts as one time). What is the total distance all the four witches have travelled?

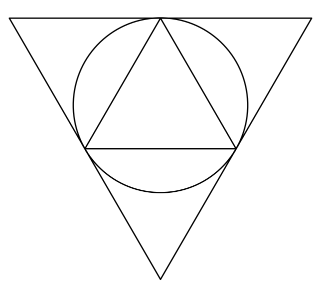
Solution. $\boxed{100.}$

We can see that they will stop when the witches with speed 3 and 4 meet for the second time. If they meet the second time after s seconds, then $4s = 3s + 10$, so $s = 10$. Then, the total distance traveled is $10 \cdot (1 + 2 + 3 + 4) = 100$.

□

Problem 1.2. Suppose A is an equilateral triangle, O is its inscribed circle, and B is another equilateral triangle inscribed in O. Denote the area of triangle T as $[T]$. Evaluate $\frac{[A]}{[B]}$.

Solution. $\boxed{4.}$



Suppose A has side length a. Since O is the inscribed circle of A, the radius r of O is $\frac{a}{2}/\sqrt{3} = \frac{a}{2\sqrt{3}}$. Since B is an equilateral triangle inscribed in O, its side length b satisfies $b = \sqrt{3}r$. Hence, $b = \frac{a}{2}$, so $\frac{[A]}{[B]} = 2^2 = 4$. □

Problem 1.3. Tim has bought a lot of candies for Halloween, but unfortunately, he forgot the exact number of candies he has. He only remembers that it's an even number less than 2020. As Tim tries to put the candies into his unlimited supply of boxes, he finds that there will be 1 candy left if he puts seven in each box, 6 left if he puts eleven in each box, and 3 left if he puts thirteen in each box. Given the above information, find the total number of candies Tim has bought.

Solution. $\boxed{666}$

Let x be the total number of candies that Tim has bought. Then, we have:

$$\begin{aligned} x &\equiv 0 \mod 2, \\ x &\equiv 1 \mod 7, \\ x &\equiv 6 \mod 11, \\ x &\equiv 3 \mod 13. \end{aligned}$$

From the last two, we must have $x \equiv 3 + 7 \cdot 13 \equiv 6 + 8 \cdot 11 = 94 \mod 143$. Combining with the first equation gives us $x \equiv 94 \mod 286$, and combining with the second equation gives us $x \equiv 94 + 2 \cdot 286 = 1 + 95 \cdot 7 \equiv 666 \mod (2 \cdot 7 \cdot 11 \cdot 13 = 1502)$. Then, since 666 is the only integer less than 2020 that is congruent to $666 \mod 2002$, we have $x = 666$. $\square$

Problem 1.4. Let $f(n)$ be a function defined on positive integers n such that $f(1) = 0$, and $f(p) = 1$ for all prime numbers p, and

$$f(mn) = nf(m) + mf(n)$$

for all positive integers m and n. Let

$$n = 277945762500 = 2^2 3^3 5^5 7^7.$$

Compute the value of $\frac{f(n)}{n}$.

Solution. $\boxed{4.}$

Let us consider the general case, where $n = p_1^{e_1} p_2^{e_2} \cdots p_k^{e_k}$. Let $S = \sum_{i=1}^k e_i$. We claim that $f(n)$ is the sum of all factors α of n such the sum of the exponents in the prime factorization of α is equal to $S-1$. For example, $f(2^2 \cdot 3 \cdot 5) = 2 \cdot 2 \cdot 3 + 2 \cdot 2 \cdot 5 + 2 \cdot 3 \cdot 5 + 2 \cdot 3 \cdot 5$. We leave the induction proof as an exercise to the reader. Then, we can rewrite $f(n)$ as

$$f(n) = \underbrace{\frac{n}{p_1} + \cdots + \frac{n}{p_1}}_{e_1 \text{ times}} + \cdots + \underbrace{\frac{n}{p_k} + \cdots + \frac{n}{p_k}}_{e_k \text{ times}} = n \sum_{i=1}^{k} \frac{e_i}{p_i}.$$

Therefore, in this case, we have $f(n) = n(\frac{2}{2} + \frac{3}{3} + \frac{5}{5} + \frac{7}{7}) = 4n$, so $\frac{f(n)}{n} = 4$.

$\square$

Problem 1.5. Compute the only positive integer value of $\frac{404}{r^2-4}$, where r is a rational number.

Solution. $\boxed{2500.}$

Let $r = \frac{a}{b}$, where a and b are relatively prime integers. Then, we have

$$\frac{404}{r^2-4} = \frac{404}{\frac{a^2}{b^2}-4} = \frac{404b^2}{a^2-4b^2} = \frac{404b^2}{(a-2b)(a+2b)}.$$

Since the greatest common factor of a, b is 1, we know that the greatest common factor of b and $a-2b$ as well as $a+2b$ is also 1. Therefore, both $a-2b$ and $a+2b$ must be factors of 404 in order for the fraction to be an integer, and no factors from either term in the denominator may be drawn from b^2.

Since $404 = 2^2 \cdot 101$, we can check all possibilities for the values of $a-2b$ and $a+2b$. In particular, we check cases where $(a-2b)(a+2b) = 4, 101,$ and 404, and solve for integers a and b. Then, we see that the only solution that works is $a-2b = 1$ and $a+2b = 101$, giving us $a = 51$ and $b = 25$, so $r = \frac{51}{25}$. Then, plugging this back into the original expression gives us

$$\frac{404}{r^2-4} = 404 \cdot \frac{625}{101} = 2500.$$

□

Problem 1.6. Let $\alpha = 3 + \sqrt{10}.$. If

$$\prod_{k=1}^{\infty}\left(1 + \frac{5\alpha+1}{\alpha^k+\alpha}\right) = m + \sqrt{n},$$

where m and n are integers, find $10m+n$.

Solution. $\boxed{50.}$

The key observation here is $\alpha^2 = 6\alpha+1$. Using this fact, we can simplify the expression:

$$\begin{aligned}\prod_{k=1}^{\infty}\left(1 + \frac{5\alpha+1}{\alpha^k+\alpha}\right) &= \prod_{k=1}^{\infty}\left(\frac{\alpha^k+6\alpha+1}{\alpha^k+\alpha}\right) \\ &= \prod_{k=1}^{\infty}\left(\frac{\alpha^k+\alpha^2}{\alpha^k+\alpha}\right) \\ &= \prod_{k=0}^{\infty}\left(\frac{\alpha^k+\alpha}{\alpha^k+1}\right).\end{aligned}$$

The product of the first $n+1$ terms of this product is

$$(1+\alpha)\left(\frac{\alpha^n}{\alpha^n+1}\right),$$

and as n grows, the fraction grows infinitely close to 1, so the product is equal to

$$\alpha + 1 = 4 + \sqrt{10}.$$

Therefore, $10m + n = 40 + 10 = 50$. □

Problem 1.7. Charlie is watching a spider in the center of a hexagonal web of side length 4. The web also consists of threads that form equilateral triangles of side length 1 that perfectly tile the hexagon. Each minute, the spider moves unit distance along one thread. If $\frac{m}{n}$ is the probability, in lowest terms, that after four minutes the spider is either at the edge of her web or in the center, find the value of $m+n$.

Solution. $\boxed{241.}$

We note that at each move the spider either moves closer to the edge, maintains its distance, or moves away from the edge. For the spider to reach the edge, it needs to move forwards four times. Not all forward moves are the same, as some forwards moves allows the spider to go to three possible forward moves while some forward moves only allow the spider to go to two possible forward moves. We will call these moves f_3 and f_2 for convenience. We see that by doing some casework that

$$\begin{aligned}
\mathbb{P}(f \to f_3 \to f_3 \to f_3) &= 1 \cdot \frac{1}{6} \cdot \frac{1}{6} \cdot \frac{1}{2} = \frac{1}{72}, \\
\mathbb{P}(f \to f_3 \to f_2 \to f_2) &= 1 \cdot \frac{1}{6} \cdot \frac{1}{3} \cdot \frac{1}{3} = \frac{1}{54}, \\
\mathbb{P}(f \to f_2 \to f_2 \to f_2) &= 1 \cdot \frac{1}{3} \cdot \frac{1}{3} \cdot \frac{1}{3} = \frac{1}{27}.
\end{aligned}$$

Now, we calculate the probability that the spider ends up back at the center. It can do so by moving back twice (after being forced to move forwards). Or moving to the side twice then moving back. We have that

$$\begin{aligned}
\mathbb{P}(f \to b \to f \to b) &= 1 \cdot \frac{1}{6} \cdot 1 \cdot \frac{1}{6} = \frac{1}{36}, \\
\mathbb{P}(f \to s \to s \to b) &= 1 \cdot \frac{1}{3} \cdot \frac{1}{3} \cdot \frac{1}{6} = \frac{1}{54}.
\end{aligned}$$

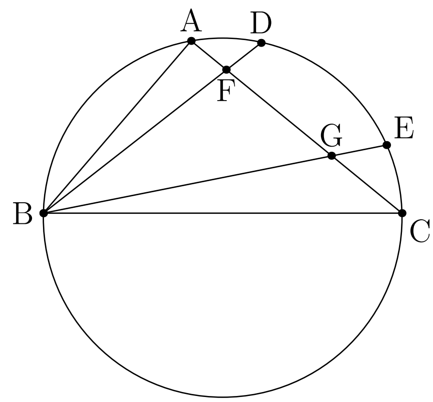
Adding all these cases together gives us the total probability of $\frac{25}{216}$, so the answer is $25+216 = 241$. $\square$

Problem 1.8. Let ABC be a triangle with $AB = 10, AC = 12$, and ω its circumcircle. Let F and G be points on $\overline{AC}$ such that $AF = 2$, $FG = 6$, and $GC = 4$, and let $\overrightarrow{BF}$ and $\overrightarrow{BG}$ intersect ω at D and E, respectively. Given that AC and DE are parallel, what is the square of the length of BC?

Solution. $\boxed{250.}$

Denote $x = BC$. Since $ACED$ is an isosceles trapezoid, we may put $y = AE = CD$. Finally, let $p = BF$, $q = DF$, $u = BG$, and $v = GE$. Note that $\angle BAC$ and $\angle BDC$ are inscribed in the same circle, so they have the same measure. Therefore, $\triangle ABF$ and $\triangle DCF$ are similar, so

$$\frac{DF}{AF} = \frac{CD}{AB} = \frac{CF}{BF} \implies \frac{q}{2} = \frac{y}{10} = \frac{10}{p}.$$



Similarly (pun intended), we have that $\triangle BCG$ and $\triangle AEG$ are similar, so we have

$$\frac{AE}{BC} = \frac{EG}{CG} = \frac{AG}{BG} \implies \frac{y}{x} = \frac{v}{4} = \frac{8}{u}.$$

Lastly, since $AC \| DE$, we have

$$\frac{p}{q} = \frac{u}{v},$$

so combining all of the above gives us

$$\frac{p}{q} = \frac{\frac{100}{y}}{\frac{y}{5}} = \frac{\frac{8x}{y}}{\frac{4y}{x}},$$

so $500 = 2x^2$, and $x^2 = 250$. $\square$

Problem 1.9. Two blue devils and 4 angels go trick-or-treating. They randomly split up into 3 non-empty groups. Let p be the probability that in at least one of these groups, the number of angels is nonzero and no more than the number of devils in that group. If $p = \frac{m}{n}$ in lowest terms, compute $m + n$.

Solution. $\boxed{76.}$

There are three ways to partition 6 into 3 groups: $(4,1,1), (3,2,1)$, and $(2,2,2)$. In the first case, there are a total of $\binom{6}{2} = 15$ ways to make the groups. To satisfy the criteria, the two devils must be in the group of 4, hence $\binom{4}{2} = 6$ groupings. In the second case, there are a total of $6 \cdot \binom{5}{2} = 60$ ways to make the groups. To satisfy the criteria, either the two devils are in the group of 3, or there is exactly one devil in the group of 2. There are $4 \cdot \binom{3}{2} + 2 \cdot 4 \cdot \binom{4}{1} = 44$ groupings. In the last case, there are $\frac{\binom{6}{2}\binom{4}{2}}{3!} = 15$ total ways to make the groups. To satisfy the criteria, the two devils cannot be in the same group, giving us $\frac{\binom{4}{2}}{2!} = 3$ bad groupings, so 12 groups that work. This gives us a total probability of $\frac{6+44+12}{15+60+15} = \frac{62}{90} = \frac{31}{45}$, so the answer is $31 + 45 = 76$.

$\square$

Problem 1.10. We know that

$$2^{22000} = \underbrace{4569878\ldots229376}_{6623 \text{ digits}}.$$

For how many positive integers $n < 22000$ is it also true that the first digit of 2^n is 4?

Solution. $\boxed{2132}$

If the first digit of a k-digit number N is c, then $c10^k \le N < (c+1)10^{k-1}$. This implies that $2c10^{k-1} \le 2N < (2c+2)10^{k-1}$, i.e. the first digit of $2N$ is at least the first digit of $2c$ and at most the first digit of $2c+1$. We apply this to the first digits of powers of two: Having a power of two with the first digit equal to 1, there are these five possibilities for the first digits of the following powers of two: (1) 1,2,4,8,1; (2) 1,2,4,9,1; (3) 1,2,5,1; (4) 1,3,6,1; (5) 1,3,7,1.

Let k be a non-negative integer such that $2k$ begins with 1 and has d digits. Then, there is a unique power of two beginning with 1 and having $d+1$ digits, and it is either 2^{k+3} (if we are in one of the situations (3), (4), (5) above) or 2^{k+4} (given that the case (1) or (2) occurs). As 2^0 (having 1 digit) and 2^{21998} (having 6623 digits) begin with 1, we can compute how many times (1) or (2) occurs when computing successive powers of two: It is exactly $21998 - 3 \cdot 6622 = 2132$ times.

Finally, observe that the case (1) and (2) are precisely those giving rise to a power of two starting with 4, therefore there are exactly 2132 such numbers in the given range.

□

2 Team Problems

Problem 2.1. At Duke, $\frac{1}{2}$ of the students like lacrosse, $\frac{3}{4}$ like football, and $\frac{7}{8}$ like basketball. Let p be the proportion of students who like at least all three of these sports and let q be the difference between the maximum and minimum possible values of p. If q is written as $\frac{m}{n}$ in lowest terms, find the value of $m+n$.

Solution. $\boxed{11.}$

The maximum occurs when the $\frac{1}{2}$ that like lacrosse alsoo like football and basketball, so the maximum is $\frac{1}{2}$. To find the minimum, note that the minimum amount that like both lacrosse and football is $\frac{1}{4}$, so we want the minimal overlap between this $\frac{1}{4}$ and the $\frac{7}{8}$ basketball lovers, which is $\frac{1}{8}$ of the student population. Thus, $q = 1 - \frac{1}{8} = \frac{3}{8}$, giving the final answer of 11.

□

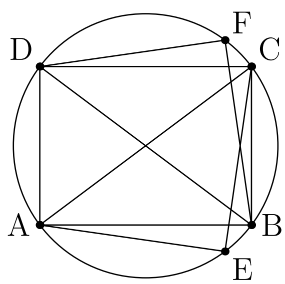
Problem 2.2. A *dukie* word is a 10-letter word, each letter is one of the four D, U, K, E such that there are four consecutive letters in that word forming the letter $DUKE$ in this order. For example, $DUDKDUKEEK$ is a dukie word, but $DUEDKUKEDE$ is not. How many different dukie words can we construct in total?

Solution. $\boxed{28576}$

First, we count the number of dukie words with at least one $DUKE$ present. We can see that there are $10-4+1=7$ possible positions for the word $DUKE$, and for the remaining 6 positions, there are 4^6 ways to choose the letter, so there are $7\cdot 4^6$ dukie words with at least one $DUKE$ present. Now we count the number of dukie words with two $DUKE$ presences. We can treat the word $DUKE$ as one "super letter", so for a word with 2 $DUKE$s present, there are 2 remaining positions, each of which have 4 letter choices. Then, we have two letters and two super letters, giving us $4^2\cdot\binom{4}{2}$ dukie words with 2 $DUKE$s present. Thus, the total number of dukie words is $7\cdot 4^6-4^2\cdot\binom{4}{2}=28576$. $\square$

Problem 2.3. Rectangle $ABCD$ has sides $AB=8$, $BC=6$. $\triangle AEC$ is an isosceles right triangle with hypotenuse AC and E above AC. $\triangle BFD$ is an isosceles right triangle with hypotenuse BD and F below BD. Find the area of $BCFE$.

Solution. $\boxed{7}$



Apply Ptolemy's Theorem on $AEBC$ to get $(10)(EB)+(5\sqrt{2})(6)=(5\sqrt{2})(8)$, so $EB=\sqrt{2}$. Applying Ptolemy's again on $EBCF$ gives us $(EF)(6)+(\sqrt{(2)})^2=(5\sqrt{2})^2$, so $EF=8$. Since $EBCF$ is isosceles, the distance from E to AB is 1, so by the Pythagorean Theorem, the height is 1. The area is therefore $\frac{6+8}{2}\cdot 1=7$. $\square$

Problem 2.4. Chris is playing with 6 pumpkins. He decides to cut each pumpkin in half horizontally into a top half and a bottom half. He then pairs each top-half pumpkin with a bottom-half pumpkin, so that he ends up having six "recombinant pumpkins". In how many ways can he pair them so that only one of the six top-half pumpkins is paired with its original bottom-half pumpkin?

Solution. $\boxed{264.}$

There are 6 ways to choose which of the 6 pumpkins is restored correctly. The other five are deranged (all halves paired incorrectly). If D_n denotes the number of derangements for n pairs of objects, we know that $D_n=(n-1)(D_{n-1}+D_{n-2})$, where $D_1=0$ and $D_2=1$ (the proof of this is left as an exercise to the reader). Then, we have $D_5=44$, so there are $6\cdot 44=264$ ways to pair the pumpkins so that only one of the pumpkins is correctly restored. $\square$

Problem 2.5. Matt comes to a pumpkin farm to pick 3 pumpkins. He picks the pumpkins randomly from a total of 30 pumpkins. Every pumpkin weighs an integer value between 7 to 16 (including 7 and 16) pounds, and there're 3 pumpkins for each integer weight between 7 to 16. Matt hopes the weight of the 3 pumpkins he picks to form the length of the sides of a triangle. Let $\frac{m}{n}$ be the probability, in lowewst terms, that Matt will get what he hopes for. Find the value of $m+n$

Solution. $\boxed{8003.}$

We compute the complement: the three weights do not form a triangle. The triplets for which this happens are: $(7,7,14),(7,7,15),(7,7,16),(7,8,15),(7,8,16),(7,9,16),(8,8,16)$. For the triplets of the form (a,a,b), there are $\binom{3}{2}\cdot 3=9$ combinations of the pumpkins, and for the triplets of the form (a,b,c), there are $3^3=27$ combinations of the pumpkins. Therefore, the complement is $9\cdot 4+27\cdot 3=117$, so the desired probability is

$$1-\frac{117}{\binom{30}{3}}=\frac{3943}{4060}.$$

Hence, the answer is $3943+4060=8003$.

□

Problem 2.6. Let a,b,c,d be distinct complex numbers such that $|a|=|b|=|c|=|d|=3$ and $|a+b+c+d|=8$. Find $|abc+abd+acd+bcd|$.

Solution. $\boxed{72.}$

Note that

$$|abc+abd+acd+bcd|=|abcd|\left|\frac{1}{a}+\frac{1}{b}+\frac{1}{c}+\frac{1}{d}\right|,$$

since magnitudes are distributive over multiplication. The trick is to express $\frac{1}{z}$ as $\frac{\overline{z}}{|z|^2}$, and to note that $|z|=|\overline{z}|$. Then, we have:

$$\begin{aligned}|abcd|\left|\frac{1}{a}+\frac{1}{b}+\frac{1}{c}+\frac{1}{d}\right| &= |a||b||c||d|\left|\frac{\overline{a}}{|a|^2}+\frac{\overline{b}}{|b|^2}+\frac{\overline{c}}{|c|^2}+\frac{\overline{d}}{|d|^2}\right| \\ &= 3^4\left|\frac{\overline{a}}{9}+\frac{\overline{b}}{9}+\frac{\overline{c}}{9}+\frac{\overline{d}}{9}\right| \\ &= 9|\overline{a}+\overline{b}+\overline{c}+\overline{d}| \\ &= 9\left|\overline{a+b+c+d}\right| \\ &= 9|(a+b+c+d)| = 9\cdot 8 = 72.\end{aligned}$$

□

Problem 2.7. A board contains the integers $1,2,\ldots,10$. Anna repeatedly erases two numbers a and b and replaces it with $a+b$, gaining $ab(a+b)$ lollipops in the process. She stops when there is only one number left in the board. Assuming Anna uses the best strategy to get the maximum number of lollipops, how many lollipops will she have?

Solution. $\boxed{54450.}$

After replacing a and b with $a+b$, Anna will gain $ab(a+b) = \frac{(a+b)^3-a^3-b^3}{3}$ lollipops. Therefore, when the numbers $a+b$ and c and replaced with $a+b+c$, Anna will gain $\frac{(a+b+c)^3-(a+b)^3-c^3}{3}$, and combined with the first quantity, results in an overall net gain of $\frac{(a+b+c)^3-a^3-b^3-c^3}{3}$. Thus, we can see that at the end, Anna will have

$$\frac{(1+2+\cdots+10)^3-1^3-2^3-\cdots-10^3}{3} = 54450$$

lollipops.

□

Problem 2.8. Ajay and Joey are playing a card game. Ajay has cards labelled $2, 4, 6, 8$, and 10, and Joey has cards labelled $1, 3, 5, 7, 9$. Each of them takes a hand of 4 random cards and picks one to play. If one of the cards is at least twice as big as the other, whoever played the smaller card wins. Otherwise, the larger card wins. Ajay and Joey have big brains, so they play perfectly. If $\frac{m}{n}$ is the probability, in lowest terms, that Joey wins, find $m+n$.

Solution. $\boxed{19.}$

First note that 1 beats everything, so if Joey has it in his hand, then he will always play it and win. Thus, we just need to consider the case when Joey doesn't draw the 1. Also note that because of this, Ajay will play assuming Joey doesn't draw the 1, because it is the only way that Ajay can win.

Note that 3 beats every card except the 4, while 9 beats only beats a 6 and 8, so playing the 9 is strictly worse than playing the 3. Thus, Joey will never play the 9, and Ajay knows this, so Ajay will play assuming Joey will play one of 3, 5, or 7.

Now, we look at Ajay's options. Both 2 and 8 beat 5 and 7, while 6 and 10 only beat 5 and 7, respectively. Thus, Ajay will never play 6 and 10, since they are strictly worse than both 2 and 8. We can further simplify by noticing that 5 and 7 are equivalent for Joey, since both beat 4 but lose to 2 and 8, and 2 and 8 are equivalent for Ajay.

Thus, if we let p be the probability that Ajay chooses 4 when he has a 4 in his hand, we have that overall the probability of him playing 4 is $.8 \cdot p$, so the probability of playing 2 or 8 is $1-0.8p$. To ensure that Joey doesn't gain an advantage, these two must be equal, so we set $0.8p = 1-0.8p$, or $p = \frac{5}{8}$, and to ensure that Ajay doesn't gain an advantage, Joey picks 3 with probability $\frac{1}{2}$ and 5 or 7 with probability $\frac{1}{2}$. Therefore, Joey will win with probability

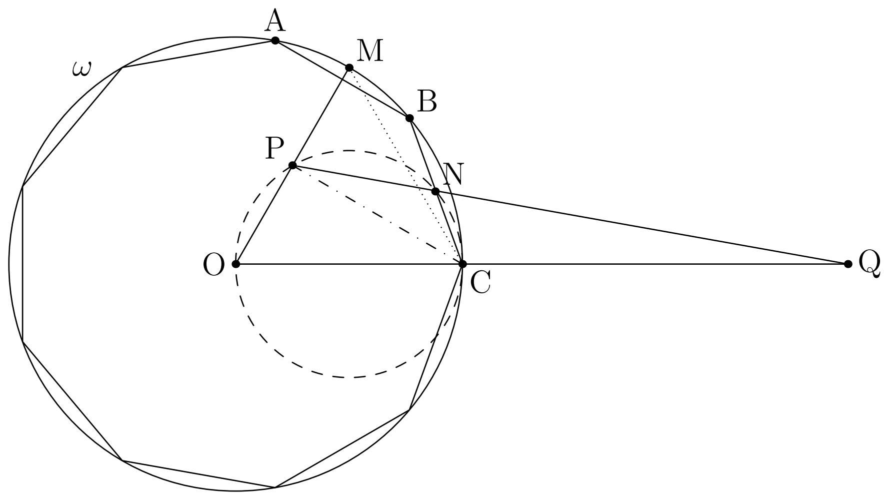
$$\frac{4}{5} + \frac{1}{5} \cdot \frac{1}{2} = \frac{9}{10},$$

so our final answer is $9+10 = 19$.

□

Problem 2.9. Let $ABCDEFGHI$ be a regular nonagon with circumcircle ω and center O. Let M be the midpoint of the shorter arc AB of ω, P be the midpoint of MO, and N be the midpoint of BC. Let lines OC and PN intersect at Q. Find the measure of $\angle NQC$ in degrees.

Solution. $\boxed{10.}$



Since C and M lie on ω, we have $OC = OM$, and $\angle MOC = \angle MOB + \angle BOC = 20 + 40 = 60$, so $\triangle OCM$ is equilateral. Then, since P is the midpoint of OM, we have $\angle OPC = 90^\circ$. Since $\angle ONC = 90^\circ$ because N is the midpoint of BC, we have that quadrilateral $OCNP$ is cyclic. Furthermore, $\angle OCN = 180^\circ - 20^\circ - 90^\circ = 70^\circ$, so $\angle OPN = 180^\circ - \angle OCN = 110^\circ$ because $OCNP$ is cyclic. Therefore, using $\triangle OQP$, we have

$$\angle NQC = \angle PQO = 180^\circ - \angle POQ - \angle QPO = 10^\circ.$$

□

Problem 2.10. In a 30×30 square table, every square contains either a kit-kat or an oreo. Let T be the number of triples (s_1, s_2, s_3) of squares such that s_1 and s_2 are in the same row, and s_2 and s_3 are in the same column, with s_1 and s_3 containing kit-kats and s_2 containing an oreo. Find the maximum value of T.

Solution. $\boxed{120000.}$

We claim that in an $n \times n$ square table there are at most $\frac{4n^2}{27}$ such triples.

Let row i and column j contain a_i and b_j kit-kats respectively, and let R be the set of red cells. For every red cell (i, j) there are $a_i b_j$ admissible triples (C_1, C_2, C_3) with $C_2 = (i, j)$, therefore

$$T = \sum_{(i,j)\in R} a_i b_j.$$

We use the inequality $2ab \le a^2 + b^2$ to obtain

$$T \le \frac{1}{2} \sum_{(i,j)\in R} (a_i^2 + b_j^2) = \frac{1}{2} \sum_{i=1}^{n} (n - a_i) a_i^2 + \frac{1}{2} \sum_{j=1}^{n} (n - b_j) b_j^2.$$

This is because there are $n - a_i$ red cells in row i and $n - b_j$ red cells in column j. Now we maximize the right-hand side.

By the AM-GM inequality we have

$$(n-x)x^2 = \frac{1}{2}(2n-2x) \cdot x \cdot x \leq \frac{1}{2}\left(\frac{2n}{3}\right)^3 = \frac{4n^3}{27},$$

with equality if and only if $x = \frac{2n}{3}$. By putting everything together, we get

$$T \leq \frac{n}{2}\frac{4n^3}{27} + \frac{n}{2}\frac{4n^3}{27} = \frac{4n^4}{27}.$$

If $n = 30$, then any coloring of the square table with $x = \frac{2n}{3} = 20$ kit-kats in each row and column attains the maximum as all inequalities in the previous argument become equalities. For example, let a cell (i, j) contain a kit-kat if i $-$ j $\equiv$ 1, 2, . . . , 20 (mod 30), and red otherwise.

Therefore the maximum value T can attain is $T = \frac{4 \cdot 30^4}{27} = 120000$. $\square$

3 Devil Round

Estimate to the nearest integer (unless specificed otherwise) the following values:

Problem 3.1. Total time (in minutes) it would take to watch all of Leonardo DiCaprio's movies.

Answer. 5621. □

Problem 3.2. Square root of the 10312020th prime number.

Answer. 13615. □

Problem 3.3. Geometric mean of the total gross of the top 20 grossing horror movies in 2020.

Answer. 1914754. □

Problem 3.4. Age (in years) of the oldest structure still in operation at Duke University.

Answer. 128. The oldest structure still in operation at Duke University is Epworth House, which opened in 1892. □

Problem 3.5. Let $V = 100$. You flip a fair coin 20 times. For each flip, if you flip a heads, then you add 20 to V, and if you flip a tails, then V becomes $\frac{1}{V}$. For example, after flipping a heads then a tails, V is $\frac{1}{120}$. Calculate the expected value of V at the end of 20 flips to three decimal places.

Answer. 40.524. □

Problem 3.6. Number of 3-pointers Zion Williamson made during his time with Duke.

Answer. 24. □

Problem 3.7. Distance, in miles, from Duke University (Durham, NC) to Duke Kunshan University (Suzhou, China), if one was to travel by foot through the Bering Strait (located between Alaska and Russia). Assume you must walk on land and that there are no travel restrictions.

Answer. 8917. □

Problem 3.8. The number of digits past the decimal point after which the digits "2020" first appear consecutively in the decimal expansion of π.

Answer. 7377. □

Problem 3.9. Total number of employees at Duke University.

Answer. 42479. □

Problem 3.10. Total number of downloads of the game Among Us.

Answer. 74000000. □