Solution Booklet

DMM 2021

1 Power Round

The theme is *Error Correction Codes*. The are a total of 50 points for this round.

1.1 Check digit

Consider the typical credit card number, which has 16 digits. Let d_i denote the ith digit. The first 15 digits is the account number, and the last digit is the *check digit*, which is given by the remainder of the sum of the first 15 digits, i.e.

$$d_{16} = d_1 + d_2 + \ldots + d_{15} \pmod{10}.$$

For example, a valid credit card number could be 1234 1234 1234 1236. If we have made a mistake in a digit and instead typed 7234 1234 1234 1236, the computer can easily check that the sum of the first 15-digits is $2 \not\equiv 6 \pmod{10}$ and detect an error.

Problem 1: (7 points total)

(a) (2 points) Describe all the valid card numbers of the form 2021 1106 _ _42 1337.

By computation we obtain $d_9 + d_{10} \equiv 1 \pmod{10}$. Hence there are 10 valid combinations where $(d_9, d_{10}) = (1, 0), (2, 9), (3, 8), \ldots, (0, 1)$.

(b) (2 points) Suppose we are given an invalid credit card number where we know that exactly one of the sixteen digits is wrong. Are we able to recover the correct credit card number? Prove your answer.

No. consider $A =$ 0000 0000 0000 0000 and $B =$ 1000 0000 0000 0001. If we are given the number 0000 0000 0000 0001, we cannot tell if it was originally A or B.

(c) (3 points) We can also consider a general formula for the check digit. Let a_i be an integer from 1 to 10. We can choose the following formula for the check digit:

$$d_{16} \equiv a_1 d_1 + a_2 d_2 + \ldots + a_{15} d_{15} \pmod{10}.$$

Find the number of tuples $(a_1, a_2, \ldots, a_{15})$ that we can choose such that an error is always detected if one of the digits is wrong.

4^{15}. We must choose a_i that is coprime to 2 or 5. There are 4 choices for each a_i.

1.2 Error Correction Codes

Consider a set-up that involve binary strings or bitstrings, which are finite sequences of the digits 0 and 1. Bob and Dylan wants to send messages to each other, but they can only do so via bitstrings.

Suppose Bob wants to send the letter A or B to Dylan, via the encoding

$$A = 0, B = 1.$$

During transmission, each bit has a probability $p = 0.1$ of changing due to random noise, and this is independent for each bit. If Bob wants to send A to Dylan using the string 0, there is a 10% chance that Dylan receives the string 1 and misinterprets the message as B.

Instead, Bob and Dylan can agree on an encoding scheme H that allows them to send a letter A or B via the following bitstrings:

$$A = 000, B = 111.$$

Bob sends the bitstring 000 or 111 to Dylan. When Dylan receives the bitstring, he checks whether it is more likely to be A or B.

For example if Dylan receives the bitstring 010, he can tell that Bob has most likely sent the letter A. This is because it will take two errors to modify 111 into 010, which is less likely to happen. Hence we have designed a code that is safe if there is at most one bit of error during transmission.

Problem 2: (2 points) Using H, what is the probability that the correct message is received by Dylan? Give the numeric answer.

The message is correct if there is at most one error. This happens with probability $(1-p)^3 + 3p(1-p)^2 = 0.972$

For any encoding scheme, the set of bitstrings that can be sent are called *codewords*. For H, the codewords are $\{000, 111\}$. Define the *distance* between two strings X and Y to be $d(X, Y) =$ total number of positions where the bit differs. Eg. $d(00, 01) = 1$.

For any encoding scheme, if we receive S, we choose the codeword that has the minimum distance from S. If there are two codewords with the same minimum distance from S, then there is no valid decoding.

Consider the encoding scheme H' given by

$$A = 00011, B = 11111, C = 00000.$$

Problem 3: (6 points total)

(a) (1 point) Under H', suppose that the bitstring X is decoded as C. What are all the possible values of $d(00000, X)$? Prove you answer.

d can be $0, 1, 2$, for $X = 00000, 10000, 11000$. If $d \geq 3$ then X will be decoded as A or B.

(b) (2 points) Find all the bitstrings X where we cannot decide on a valid decoding of X.

There are 8 such bitstrings given by

$$d(X, A) = d(X, C) : (00001, 00010, 10001, 10010, 01001, 01010, 00101, 00110)$$

A tie cannot occur between B and C, nor can it occur between B and A.

(c) (3 points) What is the probability that Bob sends A, but Dylan decodes the bitstring as B?

Changing any of the last two digits of 00011 increases the distance to A and B by 1. Hence at least two of the first three digits must be modified.

If two of the first three digits are changed to 1, then we can have at most one change in the last two digits. The probability in this case is $3p^2(1-p)(1-p^2)$

If all three digits are changed to 1, then this string will always be decoded as B. This has probability p^3. In total, we have $p^3 + 3p^2(1-p)(1-p^2) = 0.02773$.

1.3 A Larger Code

Consider a general encoding scheme. The bitstrings all have length N, and there are a total of R codewords. The minimum distance between any two codewords is denoted by D.

(For H, we have $N = 3, R = 2, D = 3$.)

Problem 4: (7 points total)

(a) (2 points) Prove the following triangle inequality: For any three codewords X, Y, Z we have

$$d(X, Y) \leq d(X, Z) + d(Y, Z)$$

If the ith digit of X and Y are different, then the ith digit of Z must be different from that of X or Y.

(b) (2 points) Define $M = \lfloor \frac{D-1}{2} \rfloor$. Conclude from earlier that if X is a codeword and $d(X, S) \leq M$, then S must be decoded as X.

Suppose otherwise, then for some other codeword $Y \neq X$ we also have $d(Y, S) \leq M$. Then $d(X, Y) \leq 2M \leq D - 1$, a contradiction.

(c) (3 points) Prove that for any encoding scheme, we have

$$R \leq \frac{2^N}{\sum_{t=0}^{M} \binom{N}{t}}.$$

By 4(b), for each codeword X there are at least $\sum_{t=0}^{M} \binom{N}{t}$ bitstrings that decodes to X. Hence there are at least $R \cdot (\sum_{t=0}^{M} \binom{N}{t})$ bitstrings with a valid decoding, and this is bounded by the total number of possible bitstrings 2^N.

We will now consider an encoding scheme J that allows us to send any 4-letter word made up of A and B. Convert the word w to a bitstring $\overline{x_1x_2x_3x_4}$ by changing $A \to 0, B \to 1$. To this bitstring, we add 3 more digits x_5, x_6, x_7 defined by

$$x_5 = x_1 + x_2 + x_4 \pmod 2, x_6 = x_1 + x_3 + x_4 \pmod 2, x_7 = x_2 + x_3 + x_4 \pmod 2.$$

Thus J is a scheme where $N = 7$ and $R = 16$.

Problem 5: (7 points total)

(a) (3 points) Prove that the distance between any two codewords in J cannot be 1 or 2.

For any two codewords, there must be at least one difference in the first 4 digits. If the number of differences in the first 4 digits is exactly 1, then two or three of the last 3 digits must be different. If the number of differences is exactly 2, then at least one of x_5, x_6, x_7 changes by exactly 1.

(b) (4 points) Show that $D = 3$, and prove that every bitstring of length 7 has a decoding.

$D = 3$ can be shown by the codewords $0000000, 1000110$. Hence for each codeword X, there are at least $1 + 7 = 8$ bitstrings that decode to X. Hence there are at least $8 * 16 = 128$ bitstrings with valid decodings, which must include all bitstrings of length 7.

If there is at most one erroneous bit in X, we can find the correct decoding of the bitstring X. This can be done in a tedious way by comparing X with each codeword. However, we would like to consider a more efficient method to decode X. Consider the three numbers $y_1, y_2, y_3 \in \{0, 1\}$ given by

$$\begin{aligned} y_1 &\equiv x_1 + x_2 + x_4 + x_5 \pmod 2, \\ y_2 &\equiv x_1 + x_3 + x_4 + x_6 \pmod 2, \\ y_3 &\equiv x_2 + x_3 + x_4 + x_7 \pmod 2. \end{aligned}$$

Problem 6: (4 points total)

(a) (1 point) Compute (y_1, y_2, y_3) for the codeword representing $BAAA$ if there is no bit error, and also if we change the 3rd bit.

The codeword for $BAAA$ is 1000110. We have $(y_1, y_2, y_3) = (0, 0, 0)$. If the $3rd$ bit is wrong, the bitstring becomes 1010110 and $(y_1, y_2, y_3) = (0, 1, 1)$

(b) (3 points) Suppose that there is at most one erroneous bit in some random bitstring X. Show that we can deduce which bit has been changed (or none at all) given only the values of y_1, y_2, y_3.

Note that by plugging in the formulas for x_5, x_6, x_7, we can show that $(y_1, y_2, y_3) = (0, 0, 0)$ when there is no errors. Next, if exactly one of $y_i = 1$, then the error must be in bit x_{i+4}. If exactly one of $y_i = 0$, we have $i = 1 \implies x_3$ is wrong, $i = 2 \implies x_2$ is wrong, $i = 3 \implies x_1$ is wrong. Finally if $y_1 = y_2 = y_3$ then x_4 is wrong.

1.4 Generic Codes

In real life, we may need to encode very long messages, and hence we have to construct larger encoding schemes. For this section you will explore the limits of what kind of encodings we can construct.

Recall the inequality from Problem 4(c). We want to find positive integers R, M, N such that the equality can be achieved. In other words, we want

$$R = \frac{2^N}{\sum_{t=0}^{M} \binom{N}{t}}. \tag{1}$$

Problem 7: (3 points) Suppose that $D = 3$ throughout this problem. Find all combinations of (N, R) such that the equation in (1) is satisfied.

We have $R(N+1) = 2^N$. Hence $N = 2^k - 1$ for some positive integer k, which implies that $R = 2^{2^k-k-1}$. Thus $(N, R) = (2^k - 1, 2^{2^k-k-1})$.

Problem 8: (8 points total) Suppose that $D = 7$ throughout this problem.

(a) (3 points) Show that

$$(N^2 - N + 6)(N + 1) = 3 \cdot 2^k$$

for some positive integer $k \geq 2$. Conclude that $N + 1 = 2^l$ or $3 \cdot 2^l$ for some non-negative integer l.

The equation in (1) rewrites as $R(1 + \binom{N}{1} + \binom{N}{2} + \binom{N}{3}) = 2^N$. Hence R is a power of 2.

Multiplying 6 on both sides, we can obtain the equation $(N^2 - N + 6)(N + 1) = 3 \cdot 2^k$ for $k \geq 2$.

(b) (3 points) Show that if $l \geq 4$, then the equation in (1) has no solutions.

Suppose that $N + 1 = 2^l$. Then substituting this into $N^2 - N + 6 = 3 \cdot 2^{k-l}$, we have $2^{2l} - 3 \cdot 2^l + 8 = 3 \cdot 2^{k-l}$. If $l \geq 4$, then $3 \cdot 2^{k-l} \equiv 8 \pmod{16}$, which implies that $k - l = 3$. Thus we have $2^{2l} - 3 \cdot 2^l - 16 = 0$, which has no solutions when $l \geq 4$. A similar argument shows that $N + 1 = 3 \cdot 2^l$ also has no solutions.

(c) (2 points) Find all the pairs (N, R) that satisfy the equation in (1).

Checking $l = 1, 2, 3$, we see that the only solutions are $(N, R) = (7, 2)$ and $(23, 4096)$

The last problem concerns finding encodings that attain equality in (1).

Problem 9: (6 points) Construct an encoding for each combination (N, R) in Problem 7.

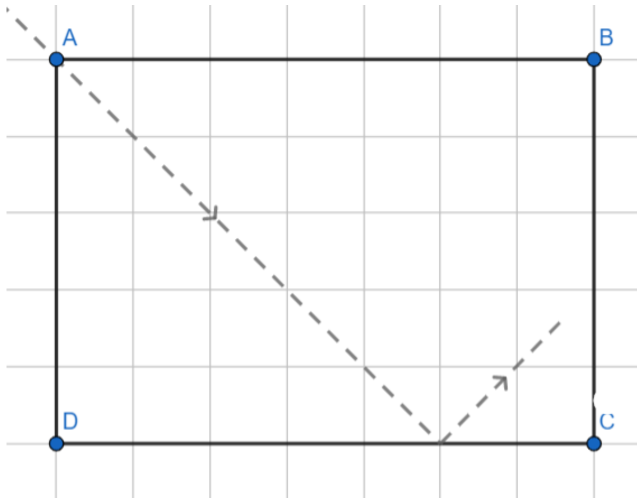
This is the Hamming code, and the idea is to extend the construction used in the scheme J.

For the bitstring of length $2^k - 1$, we designate bit $1, 2, 4, \ldots, 2^{k-1}$ to be the parity bits. The remaining $2^k - k - 1$ are used for the message.

For bit 2^i, we define it to be the sum of all the bits at position j, where the $(i+1)$th digit in the binary representation of j is 1 and $j \neq 2^i$.

2 Individual

1. There are 4 mirrors facing the inside of a 5×7 rectangle as shown in the figure. A ray of light comes into the inside of a rectangle through A with an angle of $45°$. When it hits the sides of the rectangle, it bounces off at the same angle, as shown in the diagram. How many times will the ray of light bounce before it reaches any one of the corners A, B, C, D? A bounce is a time when the ray hit a mirror and reflects off it.



Solution: $\boxed{10}$.

2. Jerry cuts 4 unit squares out from the corners of a 45×45 square and folds it into a $43 \times 43 \times 1$ tray. He then divides the bottom of the tray into a 43×43 grid and drops a unit cube, which lands in precisely one of the squares on the grid with uniform probability. Suppose that the average number of sides of the cube that are in contact with the tray is given by $\frac{m}{n}$ where m, n are positive integers that are relatively prime. Find $m + n$.

Answer: $\boxed{90}$. This can be done fairly easily with casework. Alternatively, note that each grid square has equal probability of being selected $(1/43^2)$ and every face on the box is contacted exactly once. The number of possible faces that the tray can contact is the area of the cut square, which is 2021, so the answer is just $2021/43^2 = 47/43$.

3. Compute $2021^4 - 4 \cdot 2023^4 + 6 \cdot 2025^4 - 4 \cdot 2027^4 + 2029^4$.

Solution: $\boxed{384}$. Write $x = 2025$, then we want to compute $(x-4)^4 - 4(x-2)^4 + 6x^4 - 4(x+2)^4 + (x+4)^4$. Upon expanding, all the x cancels out and we are left with 384.

4. Find the number of distinct subsets $S \subseteq \{1, 2, \ldots, 20\}$, such that the sum of elements in S leaves a remainder of 10 when divided by 32.

Solution: $2^{15} = \boxed{32768}$. For any arbitrary subset $U \subset \{1, 2, \ldots, 20\} \setminus \{1, 2, 4, 8, 16\}$, there is exactly one subset $V \subset \{1, 2, 4, 8, 16\}$ such that $U \cup V$ satisfies the conditions.

5. Some k consecutive integers have the sum 45. What is the maximum value of k?

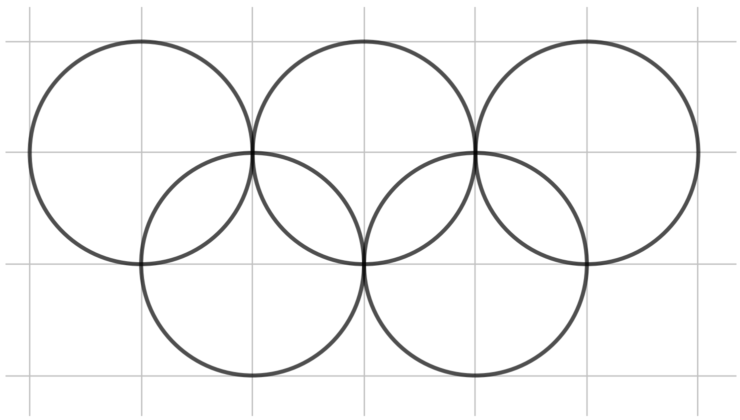
Solution: $\boxed{90}$.

Let those consecutive integers start at m and end at n, than the sum would be $m(n-m+1)+(n-m+1)(n-m)/2 = (n-m+1)(n+m)/2$. So we would have $(n-m+1)(n+m) = 90$. We have $k = n-m+1$, so its max is 90, correspond to $n = 45$ and $m = -44$.

6. Jerry picks 4 distinct diagonals from a regular nonagon (a regular polygon with 9-sides). A diagonal is a segment connecting two vertices of the nonagon that is not a side.Let the probability that no two of these diagonals are parallel be $\frac{m}{n}$ where m, n are positive integers that are relatively prime. Find $m+n$

   Solution: $\boxed{514}$. There are 9 sets of 3 diagonals that are parallel to each other. Hence the answer is $\frac{9*8*7*6*3^4}{27*26*25*24} = \frac{189}{325}$.

7. The Olympic logo is made of 5 circles of radius 1, as shown in the figure



   Suppose that the total area covered by these 5 circles is $a+b\pi$ where a, b are rational numbers. Find $10a+20b$.

   Solution: $\boxed{100}$. The total area is $5\pi - 4 \times (\frac{\pi}{2} - 1) = 4 + 3\pi$.

8. Let $P(x)$ be an integer polynomial (polynomial with integer coefficients) with $P(-5) = 3$ and $P(5) = 23$. Find the minimum possible value of $|P(-2) + P(2)|$.

   Solution: $\boxed{16}$, which is achieved by $x^2 + 2x - 12$.

   To show that it is optimal we will use the fact that $a - b | P(a) - P(b)$. We note that

$$P(2) \equiv 3 \pmod 7, P(2) \equiv 2 \pmod 3 \implies P(2) \equiv 17 \pmod{21}.$$

$$P(-2) \equiv 0 \pmod 3, P(-2) \equiv 2 \pmod 7 \implies P(-2) \equiv 9 \pmod{21}.$$

   Hence $P(-2) + P(2) \equiv 5 \pmod{21}$. Furthermore, $P(2) + P(-2)$ must be even, and so $|P(-2) + P(2)| \geq 16$.

9. There exists a unique tuple of rational numbers (a, b, c) such that the equation $a \log 10 + b \log 12 + c \log 90 = \log 2025$. What is the value of $a+b+c$?

   Solution: $\boxed{1}$. This says that $10^a 12^b 90^c = 2025$. Since a, b, c are rational, we can compare prime factors to find equations for them. These give

$$a + 2b + c = 0, b + 2c = 4, a + c = 2.$$

Solving, we obtain $(a,b,c) = (-1/2, -1, 5/2)$ and so $a+b+c = 1$.

10. Each grid of a board 7×7 is filled with a natural number smaller than 7 such that the number in the grid at the ith row and jth column is congruent to $i+j$ modulo 7. Now, we can choose any two different columns or two different rows, and swap them. How many different boards can we obtain from a finite number of swaps?

    Solution: $7! \times 6! = \boxed{3628800}$.

    The key invariant here is that for any 4 grids that form a rectangle, it splits into two pairs of opposite grids, and the sum of each pair is equal. Swapping any two rows or two columns maintains this property. So we can see that a board can be fully determined by only the information from the first rows and the first columns. So the number of boards we can obtain is the number of configurations in the first row and column.

    Now, note that every row and column is a permutation of 7 different numbers modulo 7. So, the grid $(1,1)$ has 7 choices, and there are 6! choices for the 6 other grids on the first row, and 6! choices for the 6 other grids on the first column. So in total there are $7! \times 6! = 3628800$ choices.

3 Team

1. In basketball, teams can score $1, 2$, or 3 points each time. Suppose that Duke basketball have scored 8 points so far. What is the total number of possible ways (ordered) that they have scored? For example, $(1,2,2,2,1), (1,1,2,2,2)$ are two different ways.

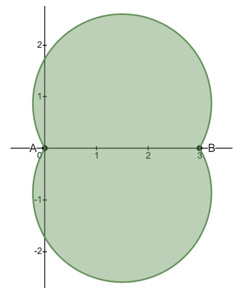
   Solution: $\boxed{81}$. We can use the recurrence $a_n = a_{n-1} + a_{n-2} + a_{n-3}$ where a_n denote the number of sequences for n points. Since $a_1 = 1, a_2 = 2, a_3 = 4$, the next few numbers are $7, 13, 24, 44, 81$.

2. All the positive integers that are coprime to 2021 are grouped in increasing order, such that the nth group contains $2n-1$ numbers. Hence the first three groups are $\{1\}$, $\{2,3,4\}$, $\{5,6,7,8,9\}$. Suppose that 2022 belongs to the kth group. Find k.

   Solution: $\boxed{44}$. From $2021 = 43 \times 47$, there are 1933 numbers up to 2022. The first n groups will include a total of n^2 numbers, and so $k = \lceil \sqrt{1933} \rceil = 44$.

3. Let $A = (0,0)$ and $B = (3,0)$ be points in the Cartesian plane. If R is the set of all points X such that $\angle AXB \geq 60^\circ$ (all angles are between 0° and 180°), find the integer that is closest to the area of R.

   Solution: $\boxed{15}$. Assume that X is in the positive y half-plane. Consider the circle with center in the positive y half-plane that passes through A and B such that the arc AB has angle 120°. Clearly, any point X satisfies the desired inequality iff it is within the circle. To handle points X in the negative y half-plane, we can reflect our work across the x-axis. Thus, R looks like the following:



From here, we can use basic geometry tricks to find that the area is $4\pi + \frac{3}{2}\sqrt{3} \approx 15.2$.

4. What is the smallest positive integer greater than 9 such that when its left-most digit is erased, the resulting number is one twenty-ninth of the original number?

   Solution: $\boxed{725}$.

   Let d be the first digit of the number, k be the number obtained after erasing the first digit, and n be the number of digits of k. Then the original number is $10^n d + k$, and the assertion can be rewritten as

$$10^n d + k = 29k,$$

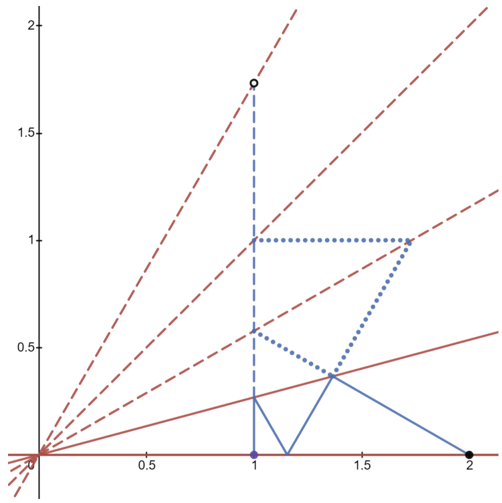
   or

$$28k = 10^n d.$$

   Since $28 = 2^2 \cdot 7$, in order for the right-hand side to be divisible by 28, the value of d must be 7, and we must also have $n \geq 2$. The choice of $n = 2$ (implying $k = 25$) gives the smallest possible number 725.

5. Jonathan is operating a projector in the cartesian plane. He sets up 2 infinitely long mirrors represented by the lines $y = \tan(15°)x$ and $y = 0$, and he places the projector at $(1, 0)$ pointed perpendicularly to the x-axis in the positive y direction. Jonathan furthermore places a screen on one of the mirrors such that light from the projector reflects off the mirrors a total of three times before hitting the screen. Suppose that the coordinates of the screen is (a, b). Find $10a^2 + 5b^2$.

   Solution: $\boxed{40}$. We wish to avoid computation as much as possible. The key insight is that, when the light from the projector hits a mirror, we can reflect the "world" instead of reflecting the light. This is conveyed in the following diagram:



The purple point is the projector, the black points are the screens with the open point as the image of the screen, the solid red lines are the mirrors, the dotted red lines are the images of the mirrors, the solid blue line is the path of the light, the dashed blue line is the image of the path of the light, and the dotted blue lines represent where the image of the light would travel if they were reflected. The image of the mirror containing the image of the screen can be represented by the line $y = \tan(60°)x$. Since the image of the light follows the line $x = 1$, we find that the image of the light hits the image of the screen at $(1, \sqrt{3})$. This point is a distance of 2 away from the origin. Since the screen is on the x-axis, the screen must be on the point $(2, 0)$.

6. Dr Kraines has a cube of size $5 \times 5 \times 5$, which is made from 5^3 unit cubes. He then decides to choose m unit cubes that have an outside face such that any two different cubes don't share a common vertex. What is the maximum value of m?

   Solution: $\boxed{26}$.

   The optimal construction for the $(2n+1)^3$ cube, is such that each face we are able to choose $(n+1)^2$ cubes, and counting together and omit the overlapping cubes, we get $6n^2 + 2$. So the answer is $6 \cdot 2^2 + 2 = 26$.

   To justify that, we can find a way to partition all the outer faces into disjoint $2\times2\times1$ blocks, $2 \times 2 \times 1$ blocks and single cubes. There are a total of $6n^2 + 2$ of them and each of them contains at most one selected cube.

7. Let $a_n = \tan^{-1}(n)$ for all positive integers n. Suppose that

$$\sum_{k=4}^{\infty} (-1)^{\lfloor \frac{k}{2} \rfloor + 1} \tan(2a_k)$$

   is equals to $\frac{a}{b}$, where a, b are relatively prime. Find $a + b$.

   Solution: $\boxed{19}$. By the tangent addition formula, we have $\tan(2a_k) = \tan(a_k + a_k) = \frac{\tan(a_k)+\tan(a_k)}{1-\tan(a_k)\tan(a_k)} = \frac{2k}{(1+k)(1-k)}$. We can use partial fractions to see that $\frac{2k}{(1+k)(1-k)} = -\frac{1}{k-1} - \frac{1}{k+1}$.

We also have that -1 when $k \equiv 0, 1 \pmod 4$ and 1 when $k \equiv 2, 3 \pmod 4$. Then, the summation telescopes to $(\frac{1}{3} + \frac{1}{4}) = \frac{7}{12}$.

8. Rishabh needs to settle some debts. He owes 90 people and he must pay $\$(101050 + n)$ to the nth person where $1 \le n \le 90$. Rishabh can withdraw from his account as many coins of values \$2021 and $\$x$ for some fixed positive integer x as is necessary to pay these debts. Find the sum of the four least values of x so that there exists a person to whom Rishabh is unable to pay the exact amount owed using coins.

   Answer: $\boxed{195}$.

   Solution a. Firstly, note that $101050 = 2021 \cdot 50$. So for Rishabh to be able to pay all of his debts, there must exist nonnegative integers a, b for each $1 \le n \le 90$ such that $2021 \cdot 50 + n = 2021a + xb$. Clearly, $a \le 50$, so we can rewrite this equation as $2021(50 - a) \equiv -n \pmod x$. If $43 \mid x$ or $47 \mid x$, then the LHS is equivalent to 0 always, so setting $n = 1$ yields that Rishabh cannot pay his debts. We now suppose that $\gcd(x, 2021) = 1$. If $x \le 51$, then it is commonly known that $2021(50 - a)$ can be equivalent to any number $\pmod x$. Thus, Rishabh can pay off his debts in this case. If $x \ge 52$, then $2021(50 - a)$ only takes on 51 values since $0 \le a \le 50$. But n spans $\min(x, 90) \ge 52$ values. Therefore, Rishabh is unable to pay off his debts. Our desired values then are $43 + 47 + 52 + 53 = 195$.

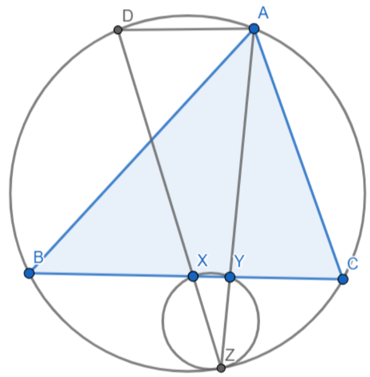
   Solution b. We proceed with the Chicken McNugget Theorem. We first note that if $43 \mid x$ or $47 \mid x$, then Rishabh can only pay in multiples of \$43 or \$47 respectively. This means he cannot pay off his debts. We then suppose that $\gcd(x, 2021) = 1$. The theorem posits that Rishabh can pay a given person if he owes them $\$(2021x - 2021 - x + 1)$ or more. If $x \le 51$, then $(2021x - 2021 - x + 1) \le 101050$ so Rishabh can pay off his debts. If $x \ge 52$, then we use a corollary that, for each integer k, exactly one of $\$k$ and $\$(2021x - 2021 - x - k)$ can be paid. For $x = 52$, we consider $k = 1924$. Since \$1924 can be paid, \$101095 cannot and Rishabh cannot pay his debts. For $x = 53$, we consider $k = 3922$ which implies that \$101117 cannot be paid. Rishabh cannot pay his debts in this case either. Since the problem asks only for the least 4 values, we can conclude that the desired values are $43 + 47 + 52 + 53 = 195$.

9. A frog starts at $(1, 1)$. Every second, if the frog is at point (x, y), it moves to $(x + 1, y)$ with probability $\frac{x}{x+y}$ and moves to $(x, y + 1)$ with probability $\frac{y}{x+y}$. The frog stops moving when its y coordinate is 10. Suppose the probability that when the frog stops its x-coordinate is strictly less than 16, is given by $\frac{m}{n}$ where m, n are positive integers that are relatively prime. Find $m + n$

   Solution: $\boxed{13}$. First show that the x-coordinate is k with probability exactly $\frac{9}{(9+k-1)(9+k)}$. This is because there are $\binom{9+k-2}{k}$ ways of reaching $(k, 10)$ as the last move must be $+1$ in y-coordinate, The probability of each path is exactly $\frac{9!\cdot(k-1)!}{(9+k)!}$. Thus the answer is $\sum_{k=1}^{15} \frac{9}{(k+8)(k+9)} = 9(\frac{1}{9} - \frac{1}{24}) = \frac{5}{8}$.

10. In the triangle ABC, $AB = 585$, $BC = 520$, $CA = 455$. Define X, Y to be points on the segment BC. Let $Z \ne A$ be the intersection of AY with the circumcircle of ABC. Suppose that XZ is parallel to AC and the circumcircle of XYZ is tangent to the circumcircle of ABC at Z. Find the length of XY.

Solution: $\boxed{64}$.



By cosine rule, we can first deduce that $\cos C = 2/7, \cos B = 2/3$. Now extend ZX to meet the circumcircle at D. By homothety, we deduce that $DACB$ is an isosceles trapezoid. Hence $ACXD$ is a parallelogram, and we have $DX = 455$. Using cosine rule, we can compute that $AD = CX = 260$, and so X is the midpoint of BC. By power of a point, $ZX = \frac{CX \cdot BX}{XD} = \frac{260^2}{455} = \frac{1040}{7}$. Finally, by $XYZ \sim DAZ$, we have $XY = AD \cdot \frac{ZX}{ZD} = \frac{260 \times 1040/7}{455 + 1040/7} = 64$.

4 Devil

1. The number of undergraduate applicants to Duke in 2020.

   Answer: 49,555, facts.duke.edu

2. The sum of the course numbers of all math courses offered at Duke this semester.

   Answer: 34,964, DukeHub registration

3. Total number of physical books in Duke libraries as of 2020.

   Answer: 7,990,426, https://library.duke.edu/about/reports-quickfacts

4. The number of total Duke alumni.

   Answer: 179,263, facts.duke.edu from Spring 2020

5. The number of views on the 5th most viewed TikTok video ever.

   Answer: 659.2 million, https://www.popbuzz.com/internet/viral/most-viewed-video-tiktok/

6. The total number of copies of Minecraft sold.

   Answer: 238,000,000, https://en.wikipedia.org/wiki/List$_o f_b est - selling_v ideo_g ames$

7. The number of characters in the longest city name.

   Answer: 58, Llanfairpwllgwyngyllgogerychwyrndrobwllllantysiliogogogoch in Wales https://www.worldatlas.com/articles/the-10-longest-place-names-in-the-world.html

8. Let the number of legal chess positions be N. Round $\ln N$ to the nearest integer.

   Answer: 103. Total positions is $4.11 \times 10^{44} through 4.85 \times 10^{44}$

   https://en.wikipedia.org/wiki/Shannon$_n umberNumber_o f_s ensible_c hess_g amesbyJohnTrump$

9. The 2021st largest prime.

   Answer: 17579

10. π^4.

    Answer: 97.40909103400242.

11. The unique integer n such that $2^n \leq 2021! \leq 2^{n+1}$.

    Answer: 19283.

12. $12^{12} - 11^{11} + 10^{10} - 9^9 + 8^8 - 7^7 + 6^6 - 5^5 + 4^4 - 3^3 + 2^2 - 1^1$

    Answer: $8 * 10^{12}$ ish, 8640417354592

13. The maximum overall team score earned at this year's Duke Math Meet.

    Answer: 118

5 Tiebreaker

1. You are standing on one of the faces of a cube. Each turn, you randomly choose another face that shares an edge with the face you are standing on with equal probability, and move to that face. Let $F(n)$ the probability that you land on the starting face after n turns. Supposed that $F(8) = \frac{43}{256}$, and $F(10)$ can be expressed as a simplified fraction $\frac{a}{b}$. Find $a+b$.

   Solution: $\boxed{1195}$

   Let's say that at the start you are standing on face 1 which is opposite of face 6, with the other faces being 2, 3, 4, 5. Define $G(n)$ to be the probability that you land on faces 2, 3, 4, or 5 after n turns. First we show that the probability you land on face 6 after n turns is also $F(n)$. This is clear because we can only reach face 6 from faces 2, 3, 4, 5, but you can reach face 1 from these faces with equal probability. It is also clear that $G(n) = \frac{1}{4}(1 - 2F(n))$ because all the probabilities must add up to 1. Furthermore, we can deduce that $F(n+1) = G(n)$. This is because to reach face 1 after $(n+1)$ turns, you must reach faces 2, 3, 4, or 5 after n turns (probability $= 4G(n)$), but from each face you have a $\frac{1}{4}$ probability of moving to 1 on the next step, so $F(n+1) = \frac{1}{4}4G(n) = G(n)$. From these observations, we can compute $F(9) = G(8) = \frac{1}{4}(1-2\times\frac{43}{256}) = \frac{85}{512}$, and $F(10) = G(9) = \frac{1}{4}(1-2\times\frac{85}{512}) = \frac{171}{1024}$, so $a+b = 1195$.