Team Round

DMM 2021

1 Team

1. In basketball, teams can score $1, 2$, or 3 points each time. Suppose that Duke basketball have scored 8 points so far. What is the total number of possible ways (ordered) that they have scored? For example, $(1, 2, 2, 2, 1), (1, 1, 2, 2, 2)$ are two different ways.

2. All the positive integers that are coprime to 2021 are grouped in increasing order, such that the nth group contains $2n - 1$ numbers. Hence the first three groups are $\{1\}$, $\{2, 3, 4\}$, $\{5, 6, 7, 8, 9\}$. Suppose that 2022 belongs to the kth group. Find k.

3. Let $A = (0, 0)$ and $B = (3, 0)$ be points in the Cartesian plane. If R is the set of all points X such that $\angle AXB \geq 60^\circ$ (all angles are between 0° and 180°), find the integer that is closest to the area of R.

4. What is the smallest positive integer greater than 9 such that when its left-most digit is erased, the resulting number is one twenty-ninth of the original number?

5. Jonathan is operating a projector in the cartesian plane. He sets up 2 infinitely long mirrors represented by the lines $y = \tan(15^\circ)x$ and $y = 0$, and he places the projector at $(1, 0)$ pointed perpendicularly to the x-axis in the positive y direction. Jonathan furthermore places a screen on one of the mirrors such that light from the projector reflects off the mirrors a total of three times before hitting the screen. Suppose that the coordinates of the screen is (a, b). Find $10a^2 + 5b^2$.

6. Dr Kraines has a cube of size $5 \times 5 \times 5$, which is made from 5^3 unit cubes. He then decides to choose m unit cubes that have an outside face such that any two different cubes don't share a common vertex. What is the maximum value of m?

7. Let $a_n = \tan^{-1}(n)$ for all positive integers n. Suppose that

$$\sum_{k=4}^{\infty} (-1)^{\lfloor \frac{k}{2} \rfloor + 1} \tan(2a_k)$$

is equals to $\frac{a}{b}$, where a, b are relatively prime. Find $a + b$.

8. Rishabh needs to settle some debts. He owes 90 people and he must pay $\$(101050 + n)$ to the nth person where $1 \leq n \leq 90$. Rishabh can withdraw from his account as many coins of values \$2021 and $\$x$ for some fixed positive integer x as is necessary to pay these debts. Find the sum of the four least values of x so that there exists a person to whom Rishabh is unable to pay the exact amount owed using coins.

9. A frog starts at $(1,1)$. Every second, if the frog is at point (x,y), it moves to $(x+1,y)$ with probability $\frac{x}{x+y}$ and moves to $(x,y+1)$ with probability $\frac{y}{x+y}$. The frog stops moving when its y coordinate is 10. Suppose the probability that when the frog stops its x-coordinate is strictly less than 16, is given by $\frac{m}{n}$ where m,n are positive integers that are relatively prime. Find $m+n$

10. In the triangle ABC, $AB = 585$, $BC = 520$, $CA = 455$. Define X,Y to be points on the segment BC. Let $Z \neq A$ be the intersection of AY with the circumcircle of ABC. Suppose that XZ is parallel to AC and the circumcircle of XYZ is tangent to the circumcircle of ABC at Z. Find the length of XY.