Individual Round

DMM 2022
**Problem 1:** Sujay sees a shooting star go across the night sky, and took a picture of it. The shooting star consists of a star body, which is bounded by four quarter-circle arcs, and a triangular tail. Suppose $AB = 2$, $AC = 4$. Let the area of the shooting star be $X$. If $6X = a - b\pi$ for positive integers $a, b$, find $a + b$.

![Diagram of shooting star]

**Your Answer:**

**Problem 2:** Assuming that each distinct arrangement of the letters in DISCUSSIONS is equally likely to occur, what is the probability that a random arrangement of the letters in DISCUSSIONS has all the S’s together?

**Your Answer:**
Problem 3: Evaluate
\[
\frac{(1 + 2022)(1 + 2022^2)(1 + 2022^4) \cdots (1 + 2022^{2^{2022}})}{1 + 2022 + 2022^2 + \ldots + 2022^{2^{2022} - 1}}.
\]

Your Answer:

Problem 4: Dr. Kraines has 27 unit cubes, each of which has one side painted red while the other five are white. If he assembles his cubes into one $3 \times 3 \times 3$ cube by placing each unit cube in a random orientation, what is the probability that the entire surface of the cube will be white, with no red faces visible? If the answer is $2^a3^b5^c$ for integers $a, b, c$, find $|a + b + c|$.

Your Answer:
Problem 5: Let $S$ be a subset of $\{1, 2, 3, \ldots, 1000, 1001\}$ such that no two elements of $S$ have a difference of 4 or 7. What is the largest number of elements $S$ can have?

Your Answer:

Problem 6: George writes the number 1. At each iteration, he removes the number $x$ written and instead writes either $4x + 1$ or $8x + 1$. He does this until $x > 1000$, after which the game ends. What is the minimum possible value of the last number George writes?

Your Answer:
**Problem 7:** List all positive integer ordered pairs \((a, b)\) satisfying \(a^4 + 4b^4 = 281 \cdot 61\).

**Your Answer:**

**Problem 8:** Karthik the farmer is trying to protect his crops from a wildfire. Karthik’s land is a \(5 \times 6\) rectangle divided into 30 smaller square plots. The 5 plots on the left edge contain fire, the 5 plots on the right edge contain blueberry trees, and the other \(5 \times 4\) plots of land contain banana bushes. Fire will repeatedly spread to all squares with bushes or trees that share a side with a square with fire. How many ways can Karthik replace 5 of his 20 plots of banana bushes with firebreaks so that fire will not consume any of his prized blueberry trees?

**Your Answer:**
Problem 9: Find $a_0 \in \mathbb{R}$ such that the sequence $\{a_n\}_{n=0}^{\infty}$ defined by $a_{n+1} = -3a_n + 2^n$ is strictly increasing.

Your Answer:

Problem 10: Jonathan is playing with his life savings. He lines up a penny, nickel, dime, quarter, and half-dollar from left to right. At each step, Jonathan takes the leftmost coin at position 1 and uniformly chooses a position $2 \leq k \leq 5$. He then moves the coin to position $k$, shifting all coins at positions 2 through $k$ leftward. What is the expected number of steps it takes for the half-dollar to leave and subsequently return to position 5?

Your Answer: