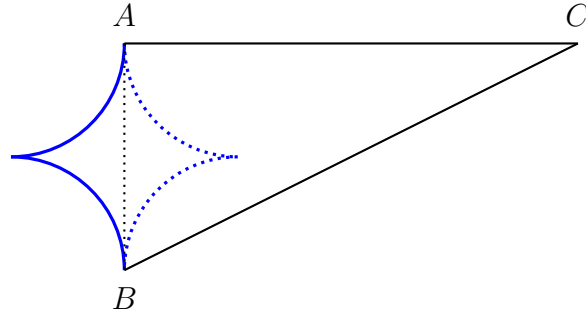


Individual Round

DMM 2022

Problem 1: Sujay sees a shooting star go across the night sky, and took a picture of it. The shooting star consists of a star body, which is bounded by four quarter-circle arcs, and a triangular tail. Suppose $AB = 2$, $AC = 4$. Let the area of the shooting star be X . If $6X = a - b\pi$ for positive integers a, b , find $a + b$.



Your Answer:

Problem 2: Assuming that each distinct arrangement of the letters in DISCUSSIONS is equally likely to occur, what is the probability that a random arrangement of the letters in DISCUSSIONS has all the S's together?

Your Answer:

Problem 3: Evaluate

$$\frac{(1 + 2022)(1 + 2022^2)(1 + 2022^4) \cdots (1 + 2022^{2^{2022}})}{1 + 2022 + 2022^2 + \cdots + 2022^{2^{2023}-1}}.$$

Your Answer:

Problem 4: Dr. Kraines has 27 unit cubes, each of which has one side painted red while the other five are white. If he assembles his cubes into one $3 \times 3 \times 3$ cube by placing each unit cube in a random orientation, what is the probability that the entire surface of the cube will be white, with no red faces visible? If the answer is $2^a 3^b 5^c$ for integers a, b, c , find $|a + b + c|$.

Your Answer:

Problem 5: Let S be a subset of $\{1, 2, 3, \dots, 1000, 1001\}$ such that no two elements of S have a difference of 4 or 7. What is the largest number of elements S can have?

Your Answer:

Problem 6: George writes the number 1. At each iteration, he removes the number x written and instead writes either $4x + 1$ or $8x + 1$. He does this until $x > 1000$, after which the game ends. What is the minimum possible value of the last number George writes?

Your Answer:

Problem 7: List all positive integer ordered pairs (a, b) satisfying $a^4 + 4b^4 = 281 \cdot 61$.

Your Answer:

Problem 8: Karthik the farmer is trying to protect his crops from a wildfire. Karthik's land is a 5×6 rectangle divided into 30 smaller square plots. The 5 plots on the left edge contain fire, the 5 plots on the right edge contain blueberry trees, and the other 5×4 plots of land contain banana bushes. Fire will repeatedly spread to all squares with bushes or trees that share a side with a square with fire. How many ways can Karthik replace 5 of his 20 plots of banana bushes with firebreaks so that fire will not consume any of his prized blueberry trees?

Your Answer:

Problem 9: Find $a_0 \in \mathbb{R}$ such that the sequence $\{a_n\}_{n=0}^{\infty}$ defined by $a_{n+1} = -3a_n + 2^n$ is strictly increasing.

Your Answer:

Problem 10: Jonathan is playing with his life savings. He lines up a penny, nickel, dime, quarter, and half-dollar from left to right. At each step, Jonathan takes the leftmost coin at position 1 and uniformly chooses a position $2 \leq k \leq 5$. He then moves the coin to position k , shifting all coins at positions 2 through k leftward. What is the expected number of steps it takes for the half-dollar to leave and subsequently return to position 5?

Your Answer: