PRUV - Summer Report

Anuk Dayaprema

July 27, 2020

Introduction

We say a three-form φ on a smooth 7-dimensional manifold M is a G_2 structure if it satisfies a certain non-degeneracy condition. The existence of such a φ is indeed equivalent to the reduction of the structure group of the frame bundle of M to G_2. In particular, φ gives rise to an orientation vol_φ and a Riemannian metric g_φ, albeit nonlinearly. Of particular interest are G_2 manifolds, which are manifolds with G_2 structure whose Riemannian holonomy group is contained in G_2. One can show that this is equivalent to the conditions $d\varphi = 0$ and $d^*_\varphi \varphi = 0$, and we say such structures are torsion-free. One idea to construct torsion-free G_2 structures, which draws from geometric flows such as the Ricci flow, is to consider the Laplacian flow

$$\frac{\partial}{\partial t}\varphi = \Delta_\varphi \varphi. \tag{1}$$

Note that torsion-free G_2 structures are critical points of the flow. Special self-similar solutions, called solitons, to the Laplacian flow for closed φ are given by triples (λ, X, φ), where $\lambda \in \mathbb{R}$ and X is a vector field, which satisfy

$$\Delta_\varphi \varphi = \lambda \varphi + \mathcal{L}_X \varphi. \tag{2}$$

Solitons give insight to singularities of the flow, which is why we consider them in this project.

Cohomogeneity-One Solitons

We say a manifold M with G_2 structure φ is cohomogeneity-one if there exists a compact Lie group G acting on M that preserves φ and the generic orbits, which we call principal orbits, are codimension one. For this project we focus on the cases where $G = SU(3)$ and $Sp(2)$. It turns out that the only cohomogeneity-one spaces where the metric induced by such a G-invariant G_2 structure is complete are the bundles of anti-self dual two forms over the Fubini-Study $\mathbb{CP}^2$ and the round S^4, respectively. Thus we are especially interested in solitons on these spaces, which must smoothly extend over the singular orbits $\mathbb{CP}^2$ and S^4. We may G-equivariantly identify the set of principal orbits with $I \times G/K$, where I is an open interval in $\mathbb{R}$ and K is the principal isotropy group, by considering the unique geodesic through a point that is orthogonal to all principal orbits. From prior work of Cleyton and Swann, any $SU(3)$-invariant G_2 structure can be written as

$$\varphi = (f_1^2\omega_1 + f_2^2\omega_2 + f_3^2\omega_3) \wedge dt + f_1 f_2 f_3(\cos\theta\alpha + \sin\theta\beta), \tag{3}$$

where f_1, f_2, f_3, θ are functions on I, with f_i nonvanishing, and ω_i, α, β are invariant differential forms on G/K. Let $X = u(t)\partial_t$ be a G-invariant vector field on $I \times G/K$. Then the soliton equation reduces to the equations

$$\frac{1}{2}(f_1^2 + f_2^2 + f_3^2) - \epsilon_\theta (f_1 f_2 f_3)' = 0 \tag{4}$$

$$\lambda f_1^2 + (f_1^2 u)' + \frac{d}{dt}\left(\frac{f_1^2}{f_2^2 f_3^2}((f_2^2 f_3^2)' - 2\epsilon_\theta f_1 f_2 f_3)\right) = 0 \tag{5}$$

$$\lambda f_2^2 + (f_2^2 u)' + \frac{d}{dt}\left(\frac{f_2^2}{f_3^2 f_1^2}((f_3^2 f_1^2)' - 2\epsilon_\theta f_1 f_2 f_3)\right) = 0 \tag{6}$$

$$\lambda f_3^2 + (f_3^2 u)' + \frac{d}{dt}\left(\frac{f_3^2}{f_1^2 f_2^2}((f_1^2 f_2^2)' - 2\epsilon_\theta f_1 f_2 f_3)\right) = 0 \tag{7}$$

$$\frac{1}{2}\left(\frac{f_1^2}{f_2^2 f_3^2}((f_2^2 f_3^2)' - 2\epsilon_\theta f_1 f_2 f_3) + \frac{f_2^2}{f_3^2 f_1^2}((f_3^2 f_1^2)' - 2\epsilon_\theta f_1 f_2 f_3)\right. \tag{8}$$

$$\left. + \frac{f_3^2}{f_1^2 f_2^2}((f_1^2 f_2^2)' - 2\epsilon_\theta f_1 f_2 f_3)\right) + \epsilon_\theta \lambda f_1 f_2 f_3 + \frac{1}{2}u(f_1^2 + f_2^2 + f_3^2) = 0, \tag{9}$$

where $\epsilon_\theta = \pm 1$. Letting τ_1, τ_2, τ_3 denote the components of the torsion two-form with respect to the frame $\{f_1^2\omega_1, f_2^2\omega_2, f_3^2\omega_3\}$, we may write the soliton equations as

$$f_1' = \frac{f_1}{2}\tau_1 + \frac{\epsilon_\theta(-f_1^2 + f_2^2 + f_3^2)}{2 f_2 f_3} \tag{10}$$

$$f_2' = \frac{f_2}{2}\tau_2 + \frac{\epsilon_\theta(f_1^2 - f_2^2 + f_3^2)}{2 f_3 f_1} \tag{11}$$

$$f_3' = \frac{f_3}{2}\tau_3 + \frac{\epsilon_\theta(f_1^2 + f_2^2 - f_3^2)}{2 f_1 f_2} \tag{12}$$

$$\tau_1' = \frac{1}{3}(-2\tau_1^2 + \tau_2^2 + \tau_3^2) + u\tau_1 + \frac{\epsilon_\theta}{f_1 f_2 f_3}(-2u f_1^2 + (f_1^2 - f_2^2 - f_3^2)\tau_1) - \frac{4}{3}\lambda \tag{13}$$

$$\tau_2' = \frac{1}{3}(\tau_1^2 - 2\tau_2^2 + \tau_3^2) + u\tau_2 + \frac{\epsilon_\theta}{f_1 f_2 f_3}(-2uf_2^2 + (-f_1^2 + f_2^2 - f_3^2)\tau_2) - \frac{4}{3}\lambda \tag{14}$$

$$\tau_3' = \frac{1}{3}(\tau_1^2 + \tau_2^2 - 2\tau_3^2) + u\tau_3 + \frac{\epsilon_\theta}{f_1 f_2 f_3}(-2uf_3^2 + (-f_1^2 - f_2^2 + f_3^2)\tau_3) - \frac{4}{3}\lambda \tag{15}$$

$$u' = \frac{1}{3}\left(\tau_1^2 + \tau_2^2 + \tau_3^2 - 3\epsilon_\theta u \frac{f_1^2 + f_2^2 + f_3^2}{f_1 f_2 f_3} - 7\lambda\right) \tag{16}$$

$$0 = f_1^2\tau_1 + f_2^2\tau_2 + f_3^2\tau_3 - 2\epsilon_\theta \lambda f_1 f_2 f_3 - u(f_1^2 + f_2^2 + f_3^2) \tag{17}$$

$$0 = \tau_1 + \tau_2 + \tau_3 \tag{18}$$

In particular, we obtain a first-order system for $(f_1, f_2, f_3, \tau_2, \tau_3)$, and we can always rescale t so that $\lambda = -1, 0,$ or 1. When $G = Sp(2)$, we simply impose the condition $f_2 = f_3$. Specifying to the $SU(3)$ case, we can show that there do not exist complete torsion-free solitons with $\lambda \neq 0$. In general, when $\lambda < 0$ we can show that positivity of u is preferred in the sense that if u becomes positive at any t_0, then it must remain positive for the rest of its existence. When φ is torsion-free, we explicitly have $u = -\frac{\lambda}{3}t$, which on the torsion-free G_2 cone (where $f_i = \frac{t}{2}$) form the one-parameter family of Gaussian solitons. When X is a gradient soliton, so $X = (df)^\sharp$, we can use the strong maximum/minimum principle to deduce that f does not attain a maximum when $\lambda \leq 0$ and does not attain a minimum when $\lambda \geq 0$.

Further Directions

Because of the nonlinearity of the soliton system, it is difficult to find explicit solutions. Thus it would be useful to obtain qualitative information about and study approximations of solutions. To this end we have two methods in mind of approaching this problem; both involve understanding better what conditions, for example initial conditions and asymptotic behavior about $t = 0$, solutions must satisfy in order to smoothly extend to the singular orbit. One idea is to mimic the stable manifold theorem from the theory of dynamical systems to obtain existence results, since the ODE system is undefined for the initial conditions imposed at the singular orbit. The other idea is to use a power-series approach as used in papers of Dr. Haskins to obtain polynomial approximate solutions. These approximations can then be used in conjunction with numerical studies, as the singularity of the soliton system near the singular orbit is problematic for many numerical ODE solvers, and also to obtain theoretical information such as degrees of freedom for initial conditions at the singular orbit.