Solution Booklet

DMM 2022

1 Power Round

The theme is *Borda Score and Elections*. There are a total of 60 points for this round. Throughout the problem, ties are broken arbitrarily (you cannot break ties to your favor).

1.1 Borda Score in Single-Winner Elections

The Duke University Math Union (DUMU) is running an election for officers! There are three voters: Alice, Bob, and Cady, and three candidates: Xavier, Yisa, and Zack. We want to select a single winner. Each voter ranks the three candidates as follows:

$$\begin{aligned} \text{Alice} &: \text{Xavier} > \text{Yisa} > \text{Zack}, \\ \text{Bob} &: \text{Zack} > \text{Xavier} > \text{Yisa}, \\ \text{Cady} &: \text{Yisa} > \text{Xavier} > \text{Zack}, \end{aligned}$$

This means, for instance, Alice prefers Xavier the most and Zack the least. In this election, one might intuitively conclude that Xavier, who has the highest average rank, should win. The DUMU executive board wants to formalize this intuition, so they decide to select the candidate with the smallest *Borda score*.

The definition for Borda score is straightforward: the Borda score of a candidate c for a voter v is simply the rank of c in v's ranking, and the Borda score of c is simply her average rank. In this example, the Borda score of Zack for both Alice and Cady are 3, and 1 for Bob. Hence, the Borda score of Zack is $(1 + 3 + 3)/3 = \frac{7}{3}$.

Problem 1: (4 points total)

(a) (2 points) Similarly compute the Borda score for Xavier and Yisa, and explain why Xavier wins under Borda score.

Solution. Borda score for Xavier is $(1+2+2)/3 = \frac{5}{3}$, and the Borda score for Yisa is $(1+2+3)/3 = 2$. Xavier wins under Borda score because he has the smallest Borda score.

(b) (2 points) If we add one more voter, is it possible for Yisa to win? Prove your answer.

Solution. Yes. If the preference of the fourth voter is Yisa > Zack > Xavier, then Yisa wins.

Going beyond this example, we explore some properties of Borda score.

Problem 2: (6 points total)

(a) (2 points) If a candidate c is ranked first by more than half of the votes in an election, does c necessarily win under Borda score? Prove your answer.

Solution. No. Consider the following election with five voters v_1, v_2, v_3, v_4, v_5 and four candidates c_1, c_2, c_3, c_4, where the preferences of the voters for the candidates are

$$\begin{aligned} v_1 &: c_1 > c_2 > c_3 > c_4, \\ v_2 &: c_1 > c_2 > c_3 > c_4, \\ v_3 &: c_1 > c_2 > c_3 > c_4, \\ v_4 &: c_4 > c_2 > c_3 > c_1, \\ v_5 &: c_4 > c_2 > c_3 > c_1. \end{aligned}$$

Then, even though c_1 is ranked first by more than half of the votes, c_2 wins the election.

(b) (2 points) Suppose c wins under Borda score in an election. If we improve the position of c in some votes and leave everything else the same (*i.e.* if we exclude c, the rankings remain the same after the change), does c still win? Prove your answer.

Solution. Yes, because the Borda score of c strictly decreases while the Borda score of other candidates won't decrease. Thus, c still wins.

(c) (2 points) Suppose c wins under Borda score in an election. We then change votes in such a way that for each vote, if a candidate w was ranked below c originally, w is still ranked below c in the new vote. Does c still win under the new votes? Prove your answer.

Solution. No. Consider the same election as in (a). If we change the votes of v_4 and v_5 such that the preferences of both voters become $c_4 > c_2 > c_1 > c_3$, then c_1 would win the election.

1.2 Borda Score in Multi-Winner Elections

More generally, let $\mathcal{V}$ denote the set of voters and $\mathcal{C}$ denote the set of candidates. Suppose there are n voters and m candidates, *i.e.* $|\mathcal{V}| = n$ and $|\mathcal{C}| = m$. Let $r_v(c)$ denote the Borda score of candidate c for voter v.

In multi-winner elections, we select a set of candidates T, which we call a *committee*, instead of a single candidate. The Borda score of T for a voter v is $r_v(T) = \min_{c \in T} r_v(c)$, and the Borda score of T is $r_{\mathcal{V}}(T) = \frac{1}{n} \sum_{v \in V} r_v(T)$. To interpret this score, for each voter, we consider the candidate with the smallest Borda score; then, we take the sum of these scores, and average it over all voters.

Problem 3: (10 points total)

(a) (2 points) Consider the following election, where we have 5 voters $\mathcal{V} = \{v_1, \ldots, v_5\}$ and 5 candidates $\mathcal{C} = \{c_1, \ldots, c_5\}$, where the preferences of the voters for the candidates are

$$\begin{aligned}
v_1 &: c_1 > c_2 > c_3 > c_4 > c_5, \\
v_2 &: c_2 > c_1 > c_4 > c_3 > c_5, \\
v_3 &: c_5 > c_2 > c_1 > c_3 > c_4, \\
v_4 &: c_3 > c_4 > c_2 > c_5 > c_1, \\
v_5 &: c_4 > c_1 > c_2 > c_3 > c_5.
\end{aligned}$$

Find the committee of size 2 with the smallest Borda score, and compute its Borda score.

Solution. The committee with the smallest Borda score is $\{c_2, c_4\}$, whose Borda score is $(2 + 1 + 2 + 2 + 1)/5 = \frac{8}{5}$.

(b) (3 points) Given an election, let T_k^* denote the committee with the smallest Borda score of size k. Is it necessarily true that $T_k^* \subset T_{k+1}^*$? Prove your answer.

Solution. No. Consider the election given in the solution for Problem 2(a). The committee of size 1 with the smallest Borda score is $\{c_2\}$. However, the committee of size 2 with the smallest Borda score is $\{c_1, c_4\}$. Hence, we do not necessarily have $T_k^* \subset T_{k+1}^*$.

(c) (5 points) If we select k candidates uniformly at random from $\mathcal{V}$ to form a committee T, what is $\mathbf{E}[r_{\mathcal{V}}(T)]$, *i.e.* the expected value of the Borda score of T? Express your answer in terms of n, m, k, and prove your answer.

Solution. We show that $\mathbf{E}[r_{\mathcal{V}}(T)] = \frac{m+1}{k+1}$. Mark $m + 1$ points on a circle. Pick a subset of $k + 1$ points uniformly at random, and then choose one point P of these $k + 1$ as the cut-off point uniformly at random. Starting from P and going clockwise, mark the next point as the candidate with rank 1, and the point after that as the candidate with rank 2, and so on, until the last point which is marked as the candidate with rank m. The picked subset comprises P and a uniformly random size-k subset of $\mathcal{C}$. By symmetry, the expected clockwise distance going from the t^{th}-smallest ranked chosen candidate to the $(t + 1)^{\text{st}}$ is the same for every $t \in \{0, 1, \ldots, k\}$, if we view P as simultaneously the 0^{th} and the $(k + 1)^{\text{st}}$ smallest. Since these $k + 1$ distances sum to $m + 1$, all of them should be $\frac{m+1}{k+1}$. In particular, we have $\mathbf{E}[r_{\mathcal{V}}(T)] = \frac{m+1}{k+1}$.

1.3 Finding a Good Committee

In practice, we often find a good committee with the following procedure: pick candidates in k rounds, during which we build sets $\emptyset = T_0 \subsetneq T_1 \subsetneq \cdots \subsetneq T_k$, and declare T_k as the selected committee. In the j^{th} round, we pick candidate $c_j \in \mathcal{C} \setminus T_{j-1}$ that minimizes $r_{\mathcal{V}}(T_{j-1} \cup \{c_j\})$. In other words, we greedily pick the candidate that minimizes the Borda score in each round. We denote this procedure by Greedy.

In this section, we explore some properties of Greedy.

Problem 4: (10 points total)

(a) (2 points) For $k = 3$, compute the committee that GREEDY produces in the election given in Problem 3(a).

Solution. The committee that GREEDY produces is $\{c_1, c_2, c_4\}$.

(b) (3 points) Does GREEDY always produce the optimal committee, *i.e.* the committee with the smallest Borda score? Prove your answer.

Solution. No. Consider the election given in the solution for Problem 2(a). The committee of size 2 with the smallest Borda score is $\{c_1, c_4\}$. However, GREEDY produces $\{c_1, c_2\}$.

(c) (5 points) Recall that T_j is the committee produced by GREEDY after j rounds, $r_v(T_j)$ is the Borda score of T_j for voter v, and $r_{\mathcal{V}}(T_j)$ is the Borda score of T_j. Prove that

$$r_{\mathcal{V}}(T_j) - r_{\mathcal{V}}(T_{j+1}) \geq \frac{\sum_{v\in\mathcal{V}} r_v(T_j)(r_v(T_j)-1)}{2n(m-j)}.$$

Solution. For a candidate $c \notin T_j$, define $\Delta_c := r_{\mathcal{V}}(T_j) - r_{\mathcal{V}}(T_j \cup \{c\})$, *i.e.* the current marginal contribution of c to the 1-Borda score. Taking the sum of Δ_c over $c \notin T_j$:

$$\sum_{c\in\mathcal{C}\setminus T_j} \Delta_c = \frac{1}{n}\sum_{v\in\mathcal{V}} \sum_{j=1}^{r_v(T_j)-1} j = \frac{\sum_{v\in\mathcal{V}} r_v(T_j)(r_v(T_j)-1)}{2n}.$$

GREEDY chooses $c^* = \arg\max_c \Delta_c$ at the $(j+1)^{\text{st}}$ iteration, giving us

$$r_{\mathcal{V}}(T_j) - r_{\mathcal{V}}(T_{j+1}) = \Delta_{c^*} \geq \frac{1}{m-j}\sum_{c\in\mathcal{C}\setminus T_j} \Delta_c = \frac{\sum_{v\in\mathcal{V}} r_v(T_j)(r_v(T_j)-1)}{2n(m-j)}.$$

In the following, we investigate theoretical guarantees on the quality of the committee produced by GREEDY. You can use the conclusion from Problem 4(c) even if you haven't solved it. Complete proofs to the Problem 5 can be hard, and partial credits will be offered to useful observations and reasonable attempts. Write down whatever you think can take you closer to the solution!

Let RAND(k) denote the answer of Problem 3(c), *i.e.* the expected Borda score of a randomly selected committee of size k. Recall that T_k is the committee produced by GREEDY after k rounds.

Problem 5: (20 points total)

(a) (15 points) Show that, for any election, we have

$$r_{\mathcal{V}}(T_k) \leq 2 \cdot \text{RAND}(k).$$

Solution. Since $\textsc{Rand}(k) = \frac{m+1}{k+1}$, we actually want to prove $r_{\mathcal{V}}(T_k) \leq 2 \cdot \frac{m+1}{k+1}$. We prove by induction. The base case clearly holds. Now suppose that the claim holds for some $k-1$ and we will prove that it also holds for k. By induction hypothesis, we have:

$$r_{\mathcal{V}}(T_{k-1}) \leq 2 \cdot \frac{m+1}{k}.$$

If $r_{\mathcal{V}}(T_{k-1}) \leq 2 \cdot \frac{m+1}{k+1}$, then $r_{\mathcal{V}}(T_k) \leq r_{\mathcal{V}}(T_{k-1}) \leq 2 \cdot \frac{m+1}{k+1}$ finishes the proof. Thus, we only need to consider the following case:

$$2 \cdot \frac{m+1}{k+1} < r_{\mathcal{V}}(T_{k-1}) \leq 2 \cdot \frac{m+1}{k}.$$

We now have the following, where the first inequality is by the conclusion from Problem 4(c) and second by Cauchy-Schwarz inequality:

$$\begin{aligned} r_{\mathcal{V}}(T_{k-1}) - r_{\mathcal{V}}(T_k) &\geq \frac{\sum_{v \in \mathcal{V}} r_v(T_{k-1})(r_v(T_{k-1}) - 1)}{2n(m-k+1)} \\ &\geq \frac{\frac{1}{n}(\sum_{v \in \mathcal{V}} r_v(T_{k-1}))^2 - \sum_{v \in \mathcal{V}} r_v(T_{k-1})}{2n(m-k+1)} \\ &= \frac{(\sum_{v \in \mathcal{V}} r_v(T_{k-1}))^2}{2n^2(m+1)} \cdot \frac{m+1}{m-k+1} \cdot \frac{\sum_{v \in \mathcal{V}}(r_v(T_{k-1}) - 1)}{\sum_{v \in \mathcal{V}} r_v(T_{k-1})}. \end{aligned}$$

Since $r_{\mathcal{V}}(T_{k-1}) \geq 2 \cdot \frac{m+1}{k+1}$ by assumption, we have:

$$\begin{aligned} \frac{m+1}{m-k+1} \cdot \frac{\sum_{v \in \mathcal{V}}(r_v(T_{k-1}) - 1)}{\sum_{v \in \mathcal{V}} r_v(T_{k-1})} &\geq \frac{m+1}{m-k+1} \cdot \frac{2 \cdot \frac{m+1}{k+1} - 1}{2 \cdot \frac{m+1}{k+1}} \\ &= \frac{2(m+1) - k - 1}{2(m+1) - 2k} \geq 1. \end{aligned}$$

Combining the previous two inequalities, we therefore have:

$$r_{\mathcal{V}}(T_{k-1}) - r_{\mathcal{V}}(T_k) \geq \frac{(\sum_{v \in \mathcal{V}} r_v(T_{k-1}))^2}{2n^2(m+1)} = \frac{r_{\mathcal{V}}^2(T_{k-1})}{2(m+1)},$$

which is equivalent to:

$$r_{\mathcal{V}}(T_k) \leq -\frac{1}{2(m+1)} r_{\mathcal{V}}^2(T_{k-1}) + r_{\mathcal{V}}(T_{k-1}).$$

Notice that the right hand side is a quadratic function in $r_{\mathcal{V}}(T_{k-1})$, which is monotonically increasing for $r_{\mathcal{V}}(T_{k-1}) \leq m+1$. Since $r_{\mathcal{V}}(T_{k-1}) \leq 2 \cdot \frac{m+1}{k} \leq m+1$, the right hand side reaches its maximum at $2 \cdot \frac{m+1}{k}$. Thus, we have:

$$r_{\mathcal{V}}(T_k) \leq -\frac{1}{2(m+1)} \cdot \left(\frac{2(m+1)}{k}\right)^2 + \frac{2(m+1)}{k} \leq \frac{2(m+1)}{k+1},$$

which concludes our induction.

(b) (5 points) Show that there exists an instance such that

$$r_{\mathcal{V}}(T_k) > \textsc{Rand}(k).$$

(Hint: You want to show that with appropriate choice of n, m, k, and rankings of voters for candidates, this inequality is possible.)

Solution. Take m to be a power of 2 that is large enough, $n = (m-1)!$, and $k = 2$. Suppose the candidates are $\{c_1, \ldots, c_m\}$. c_1 ranks at the $(\frac{m}{2})^{\text{th}}$ place for every voter, and any pair of voters have different preferences on other candidates. Greedy chooses candidate 1 in the first round, and without loss of generality, we assume that Greedy chooses candidate 2 in the second round; *i.e.* $T_k = \{c_1, c_2\}$. The Borda score of this committee is $\frac{3}{8}m$, while $\textsc{Rand}(k) = \frac{1}{3}(m+1)$. Since m is large enough, we have $r_{\mathcal{V}}(T_k) > \textsc{Rand}(k)$. in this instance.

1.4 Generalization: s-Borda Score

One commonly used generalization of Borda Score is *s-Borda score*. In this section, we use $r_{\mathcal{V}}(T)$ to denote the s-Borda score of T instead of the usual Borda score, which is defined by

$$r_{\mathcal{V}}(T) = \frac{1}{n} \sum_{v \in \mathcal{V}} \left(\min_{Q \subseteq T, |Q| = s} \sum_{c \in Q} r_v(c) \right).$$

Here, $r_v(c)$ still denotes the Borda score of c for v, which is the rank of c in v's ranking.

Problem 6: (10 points total)

(a) (3 points) Interpret this definition in plain English.

Solution. For each voter, consider the s candidates in T whose Borda score is the smallest. Now, take the sum of these scores, and average it over all the voters.

(b) (2 points) For $s = 2$, compute the committee of size 3 with the smallest s-Borda score in the election given in Problem 3(a).

Solution. The committee with the smallest s-Borda score is $\{c_1, c_2, c_4\}$.

(c) (5 points) If we select k candidates uniformly at random from $\mathcal{V}$ to form a committee T, what is $\mathbf{E}[r_{\mathcal{V}}(T)]$, *i.e.* the expected value of the s-Borda score of T? Prove your answer.

Solution. The proof for Problem 3(c) actually shows that $\mathbf{E}[r_{\mathcal{V}}(T)] = \frac{s(s+1)}{2} \cdot \frac{m+1}{k+1}$.

2 Team Round

1. The serpent of fire and the serpent of ice play a game. Since the serpent of ice loves the lucky number 6, he will roll a fair 6-sided die with faces numbered 1 through 6. The serpent of fire will pay him $\log_{10} x$, where x is the number he rolls. The serpent of ice rolls the die 6 times. His expected total amount of winnings across the 6 rounds is k. Find 10^k.

   Solution. The expected winnings for one roll of the die is
   $$\frac{1}{6}(\log_{10} 1 + \log_{10} 2 + \cdots + \log_{10} 6) = \frac{\log 6!}{6}.$$
   Hence, we have $k = \log_{10} 6!$ and $10^k = 10^{\log_{10} 6!} = 6! = \boxed{720}$.

2. Let $a = \log_3 5$, $b = \log_3 4$, $c = -\log_3 20$, evaluate $\frac{a^2+b^2}{a^2+b^2+ab} + \frac{b^2+c^2}{b^2+c^2+bc} + \frac{c^2+a^2}{c^2+a^2+ca}$.

   Solution. We can easily verify that $a+b+c = 0$. Hence, $a^2+b^2+ab = \frac{1}{2}[a^2+b^2+(a+b)^2] = \frac{1}{2}(a^2+b^2+c^2)$. Similarly, we have $b^2+c^2+bc = c^2+a^2+ca = \frac{1}{2}(a^2+b^2+c^2)$. Therefore, the required sum is $\frac{(a^2+b^2)+(b^2+c^2)+(c^2+a^2)}{\frac{1}{2}(a^2+b^2+c^2)} = \boxed{4}$.

3. Let $\triangle ABC$ be an isosceles obtuse triangle with $AB = AC$ and circumcenter O. The circle with diameter AO meets BC at points X, Y, where X is closer to B. Suppose $XB = YC = 4$, $XY = 6$, and the area of $\triangle ABC$ is $m\sqrt{n}$ for positive integers m and n, where n does not contain any square factors. Find $m + n$.

   Solution. Let M be the intersection of BC and AO. Suppose $AM = x$ and $MO = y$. Applying Pythagorean theorem on $\triangle OMC$, we get $(x + y)^2 = y^2 + 49$. Also, we have $xy = XM \cdot MY = 9$. Solving these equations gives $x = \sqrt{31}$. Hence, the area of $\triangle ABC$ is $7\sqrt{31}$, giving $m + n = \boxed{38}$.

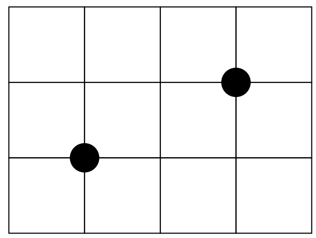
4. Alice is not sure what to have for dinner, so she uses a fair 6-sided die to decide. She keeps rolling, and if she gets all the even numbers (*i.e.* getting all of 2, 4, 6) before getting any odd number, she will reward herself with McDonald's. Find the probability that Alice could have McDonald's for dinner.

   Solution. The probability that Alice gets three evens for the first three distinct numbers is equal to the probability that she gets any three distinct numbers. Hence, the probability that Alice could have McDonald's for dinner is $\frac{1}{\binom{6}{3}} = \boxed{\frac{1}{20}}$.

5. How many distinct ways are there to split 50 apples, 50 oranges, 50 bananas into two boxes, such that the products of the number of apples, oranges, and bananas in each box are non-zero and equal?

Solution. Let $25+a$ (resp. $25+b$, $25+c$) and $25-a$ (resp. $25-b$, $25-c$) be the number of apples (resp. oranges, bananas) in box 1 and 2 respectively, where $-24 \leq a, b, c \leq 24$. We have $(25+a)(25+b)(25+c) = (25-a)(25-b)(25-c)$, which simplifies to $abc = 5^4(a+b+c)$. Hence, at least one of a, b, c must be 0. This implies $a+b+c = abc = 0$. If $a = 0$, then $b = -c$, and we have 49 choices for b and c. Similarly, we have 49 choices each for $b = 0$ and $c = 0$. The case of $(a, b, c) = (0, 0, 0)$ is triple counted, so the answer is $49 \times 3 - 2 = \boxed{145}$.

6. Sujay and Rishabh are taking turns marking lattice points within a square board in the Cartesian plane with opposite vertices $(1, 1), (n, n)$ for some constant n. Sujay loses when the two-point pattern P below shows up:



That is, Sujay loses when there exists a pair of points (x, y) and $(x+2, y+1)$. He and Rishabh stop marking points when the pattern P appears on the board. If Rishabh goes first, let S be the set of all integers $3 \leq n \leq 100$ such that Rishabh has a strategy to always trick Sujay into being the one who creates P. Find the sum of all elements of S.

Solution. We claim that Rishabh has a winning strategy for odd n only. Firstly, if n is even, then Sujay should perform the same moves as Rishabh but rotated $180°$ about the center of the region. It is clear that this is always a valid move. Since P is rotationally symmetric, Sujay can only complete the pattern if Rishabh completes the pattern right before. Thus, Sujay wins in this case.

For odd n, Rishabh should first mark the center of the region. Since the rotation of the center point is itself, Sujay cannot use his previous strategy and must arbitrarily mark a point. Rishabh can then mirror Sujay's moves and will therefore win using similar logic as in the even n case. Thus, the answer is $3 + 5 + \ldots + 99 = \boxed{2499}$.

7. Let a be the shortest distance between the origin $(0, 0)$ and the graph of $y^3 = x(6y - x^2) - 8$. Find $\lfloor a^2 \rfloor$. ($\lfloor x \rfloor$ is the largest integer not exceeding x)

Solution. From the equation, we obtain $x^3 + y^3 + 2^3 - 2 \cdot 3xy = 0$. Using the equality $a^3 + b^3 + c^3 - 3abc = (a+b+c)(a^2+b^2+c^2-ab-bc-ca)$, we get $(x+y+2)(x^2+y^2+4-2x-2y-xy) = 0$. Noting that $x^2+y^2+4-2x-2y-xy = \frac{1}{2}(x-y)^2 + \frac{1}{2}(x-2)^2 + \frac{1}{2}(y-2)^2$, we have either $x+y+2 = 0$ or $x = y = 2$. The shortest distance from $(0, 0)$ to $x+y+2 = 0$ is $\sqrt{2}$; the distance from $(0, 0)$ to $(2, 2)$ is $\sqrt{8}$. Hence, the answer is $\boxed{2}$.

8. Find all real solutions to the following equation:

$$2\sqrt{2}x^2 + x - \sqrt{1-x^2} - \sqrt{2} = 0.$$

Solution. Let $x = \sin\theta$, where $\theta \in [-\frac{\pi}{2}, \frac{\pi}{2}]$. Then, we have

$$2\sqrt{2}\sin^2\theta + \sin\theta - \cos\theta - \sqrt{2} = 0,$$

which simplifies to

$$(\sin\theta - \cos\theta)\left[2\sin\left(\theta + \frac{\pi}{4}\right) + 1\right] = 0.$$

If $\sin\theta - \cos\theta = 0$, we have $\theta = \frac{\pi}{4}$ and $x = \frac{\sqrt{2}}{2}$. If $\sin\left(\theta + \frac{\pi}{4}\right) = -\frac{1}{2}$, we have $\theta = -\frac{5}{12}\pi$ and $x = -\frac{\sqrt{6}+\sqrt{2}}{4}$. Hence, all real solutions to the given equation are $\boxed{x = \frac{\sqrt{2}}{2}}$ or $\boxed{x = -\frac{\sqrt{6}+\sqrt{2}}{4}}$.

9. Given the expression $S = (x^4 - x)(x^2 - x^3)$ for $x = \cos\frac{2\pi}{5} + i\sin\frac{2\pi}{5}$, find the value of S^2.

Solution. The given value of x is a fifth root of unity, with $x^5 = 1$. This allows us to expand S and divide out factors of x^5:

$$S = (x^4 - x)(x^2 - x^3) = x^6 - x^7 - x^3 + x^4 = x^4 - x^3 - x^2 + x.$$

Squaring this expression yields

$$S^2 = (x^4 - x^3 - x^2 + x)^2 = x^8 + x^6 + x^4 + x^2 + 2(-x^7 - x^6 + x^5 + x^5 - x^4 - x^3).$$

We again use the fact that $x^5 = 1$ to divide out factors of x^5, obtaining

$$\begin{aligned} S^2 &= x^3 + x + x^4 + x^2 + 2(-x^2 - x + 2 - x^4 - x^3) \\ &= 4 - (x + x^2 + x^3 + x^4). \end{aligned}$$

Factoring $x^5 - 1 = 0$, we obtain $(x-1)(x^4 + x^3 + x^2 + x + 1) = 0$. Division by $x - 1$ yields $x^4 + x^3 + x^2 + x = -1$, which can be plugged into the expression for S^2 to obtain

$$S^2 = 4 - (x + x^2 + x^3 + x^4) = 4 - (-1) = \boxed{5}.$$

10. In a 32 team single-elimination rock-paper-scissors tournament, the teams are numbered from 1 to 32. Each team is guaranteed (through incredible rock-paper-scissors skill) to win any match against a team with a higher number than it, and therefore will lose to any team with a lower number. Each round, teams who have not lost yet are randomly paired with other teams, and the losers of each match are eliminated. After the 5 rounds of the tournament, the team that won all 5 rounds is ranked 1st, the team that lost the 5th round is ranked 2nd, and the two teams that lost the 4th round play each other for 3rd and 4th place. What is the probability that the teams numbered 1, 2, 3, and 4 are ranked 1st, 2nd, 3rd, and 4th respectively? If the probability is $\frac{m}{n}$ for relatively prime integers m and n, find m.

Solution. The first 3 rounds of the tournament are equivalent to randomly dividing all 32 teams into four groups of 8 and selecting the best of each group to advance to the semifinals. For the top 4 teams to be ranked correctly, they all must make it to the semifinals and therefore must all be in different groups. Imagine placing the teams into the four groups in order: team 1 is placed first, so there is a $\frac{32}{32}$ chance they are placed in a different group than the others; team 2 is placed second with a $\frac{24}{31}$ chance of avoiding being in the same group as team 1; team 3 is placed third with a $\frac{16}{30}$ chance of avoiding team 1 and team 2's groups; and team 4 has a $\frac{8}{29}$ chance of being placed correctly.

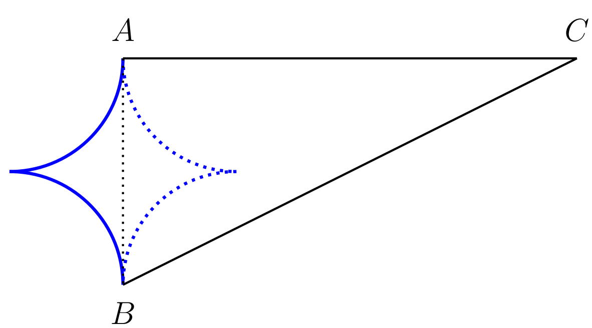
After making it to the semifinals, the top 4 teams must still be matched correctly to be ranked in the right order. Team 1 will certainly win the semifinals and finals, while team 4 will lose the semifinals and the match for 3rd place. Team 2 and 3, however, must be paired correctly to ensure that team 2 advances to the finals instead of team 3. This has a $\frac{2}{3}$ chance of happening, because team 2 will be randomly paired with team 1, 3, or 4 for the semifinals, and will make it to the finals if paired against team 3 or 4. All in all, this gives a total probability of the top 4 teams being ranked correctly of

$$\frac{24}{31} \cdot \frac{16}{30} \cdot \frac{8}{29} \cdot \frac{2}{3} = \frac{1024}{13485},$$

the numerator of which is $\boxed{1024}$.

3 Individual Round

1. Sujay sees a shooting star go across the night sky, and took a picture of it. The shooting star consists of a star body, which is bounded by four quarter-circle arcs, and a triangular tail. Suppose $AB = 2$, $AC = 4$. Let the area of the shooting star be X. If $6X = a - b\pi$ for positive integers a, b, find $a + b$.



Solution. We have $6X = 6\left(4 + 2 - \frac{\pi}{2}\right) = 36 - 3\pi$. Hence, $a + b = \boxed{39}$.

2. Assuming that each distinct arrangement of the letters in DISCUSSIONS is equally likely to occur, what is the probability that a random arrangement of the letters in DISCUSSIONS has all the S's together?

Solution. The total number of arrangements is $\frac{11!}{2!4!}$, and the number of arrangements with S's together is $\frac{8!}{2!}$. Hence, the probability that a random arrangements has all the S's together is $\frac{8!/2!}{11!/(2!\cdot 4!)} = \boxed{\dfrac{4}{165}}$.

3. Evaluate
$$\frac{(1+2022)(1+2022^2)(1+2022^4)\cdots(1+2022^{2^{2022}})}{1+2022+2022^2+\ldots+2022^{2^{2023}-1}}.$$

Solution. Multiply both the numerator and the denominator by $(2022-1)$. Observe that $(x-1)(x+1)(x^2+1)\cdots(x^{2^n}+1) = x^{2^{n+1}}-1$ and $(x-1)(1+x+x^2+\cdots+x^n) = x^{n+1}-1$, we get the answer is $\boxed{1}$.

4. Dr. Kraines has 27 unit cubes, each of which has one side painted red while the other five are white. If he assembles his cubes into one $3\times3\times3$ cube by placing each unit cube in a random orientation, what is the probability that the entire surface of the cube will be white, with no red faces visible? If the answer is $2^a3^b5^c$ for integers a, b, c, find $|a+b+c|$.

Solution. Each of the 8 corner pieces has a $\frac{1}{2}$ probability of having only white faces visible, as 3 of the 6 faces are concealed by other unit cubes. The 12 edge pieces each have a $\frac{2}{3}$ probability of having only white faces visible, as 4 of the 6 possible locations for the red face to be are concealed. Finally, the 6 pieces at the center of a side have a $\frac{5}{6}$ chance of being placed so the 1 visible face is white. This gives a total probabilty of
$$\left(\frac{1}{2}\right)^8\cdot\left(\frac{2}{3}\right)^{12}\cdot\left(\frac{5}{6}\right)^6 = \frac{5^6}{3^{18}2^2},$$
so the answer is $|6-18-2| = \boxed{14}$.

5. Let S be a subset of $\{1,2,3,\ldots,1000,1001\}$ such that no two elements of S have a difference of 4 or 7. What is the largest number of elements S can have?

Solution. First consider taking such a subset of $\{1,2,\ldots,11\}$. There can be no disadvantage to including 1 in our subset, so we begin by including 1 and eliminating 5 and 8. Now, at most one from each pair $(2,9)$, $(3,7)$, $(4,11)$, $(6,10)$ can be chosen, so no more than 5 elements can be in our subset, with the subset $\{1,3,4,6,9\}$ being a possible example of a subset containing 5 elements.

Taking the complete set from the problem, it can be divided into 91 groups of 11, from each of which a maximum of 5 elements can be taken by the logic above. This gives an upper bound of $5\cdot91 = \boxed{455}$, which is possible by taking $S = \{x \mid x\in S,\ x \bmod 11 \in \{1,3,4,6,9\}\}$.

6. George writes the number 1. At each iteration, he removes the number x written and instead writes either $4x+1$ or $8x+1$. He does this until $x>1000$, after which the game ends. What is the minimum possible value of the last number George writes?

Solution. We will consider every number George writes in binary. In binary, turning x into $4x+1$ is equivalent to concatenating 01 right after x. Similarly, turning x into $8x+1$ is equivalent to concatenating 001 right after x. Essentially, George starts with 1, and at each step, he appends either 01 or 001 right after x.

Note that it is impossible to use only 10 digits. If we use only 10 digits, the second digit of x must be 0, making it smaller than 1000. Thus, the final number must have at least 11 digits. The first digit of x must be 1, and we want to fill the rest of the 10 digits with 01 and 001. Suppose we use a 01's and b 001's. Then, we have $2a+3b=10$, the only solution of which is $a=b=2$. In order to minimize the value of x, we need to place 01's as right as possible and place 001's as left as possible. Thus, the binary representation of the final answer is 10010010101, which equals $\boxed{1173}$.

7. List all positive integer ordered pairs (a,b) satisfying $a^4+4b^4=281\cdot 61$.

Solution. By Sophie Germain, we can write

$$a^4+4b^4=((a+b)^2+b^2)((a-b)^2+b^2)=61\cdot 281.$$

Since $a,b>0$, the first term is greater than the second term so we have two cases. If $(a-b)^2+b^2=1$, then either $a-b=0,b=1$, or $a-b=1,b=0$, or $a-b=-1,b=0$. None of them satisfies the given equation, so $(a-b)^2+b^2=61$ and $(a+b)^2+b^2=281$. The only two square numbers that sum to 61 are 25 and 36, so either $a-b=5,b=6$ or $a-b=6,b=5$ which implies that $a=11,b=5,6$. Testing these two pairs on $(a+b)^2+b^2=281$ gives $(a,b)=\boxed{(11,5)}$.

8. Karthik the farmer is trying to protect his crops from a wildfire. Karthik's land is a 5×6 rectangle divided into 30 smaller square plots. The 5 plots on the left edge contain fire, the 5 plots on the right edge contain blueberry trees, and the other 5×4 plots of land contain banana bushes. Fire will repeatedly spread to all squares with bushes or trees that share a side with a square with fire. How many ways can Karthik replace 5 of his 20 plots of banana bushes with firebreaks so that fire will not consume any of his prized blueberry trees?

Solution. It is clear that each of the 5 rows must have a firebreak, and that the firebreaks must be at most 1 column away from adjacent firebreaks. Let the second and fifth columns be *edge* columns, and let the third and fourth columns be *center* columns. We proceed by recursion.

Let E_k be the number of configurations where the firebreaks in the first k rows are at most 1 columns away from each other and the kth firebreak is on an edge column. Likewise, let C_k be the number of configurations where the firebreaks in the first k rows are at most 1 columns away from each other and the kth firebreak is on a center column. We have the following recursive formula

$$E_k=E_{k-1}+C_{k-1},C_k=E_{k-1}+2C_{k-1}.$$

This is true because when we add a firebreak in the kth row and an edge column, the previous firebreak is either in an edge column or center column. When we add a firebreak in the kth row and a center column, the previous firebreak is either in an edge column or one of two center columns. Since $E_1 = C_1 = 2$, we can calculate E_5 and C_5 as follows:

k	1	2	3	4	5
E_k	2	4	10	26	68
C_k	2	6	16	42	110

Thus, our answer is $68 + 110 = \boxed{178}$. Note that the terms in the table are actually twice the terms of the Fibonacci sequence which can be proven by induction.

9. Find $a_0 \in \mathbb{R}$ such that the sequence $\{a_n\}_{n=0}^{\infty}$ defined by $a_{n+1} = -3a_n + 2^n$ is strictly increasing.

Solution. Dividing both sides of the recurrence relation by 2^{n+1} gives

$$\frac{a_{n+1}}{2^{n+1}} = -\frac{3}{2} \cdot \frac{a_n}{2^n} + \frac{1}{2}.$$

Letting $b_n = \frac{a_n}{2^n}$ gives $b_{n+1} = -\frac{3}{2}b_n + \frac{1}{2}$, which is equivalent to $b_{n+1} - \frac{1}{5} = -\frac{3}{2}\left(b_n - \frac{1}{5}\right)$. Hence, we get $b_n - \frac{1}{5} = (b_0 - \frac{1}{5})(-\frac{3}{2})^n = (a_0 - \frac{1}{5})(-\frac{3}{2})^n$. Therefore, we have $a_n = 2^n \cdot b_n = (-3)^n[(a_0 - \frac{1}{5}) + \frac{1}{5}(-\frac{2}{3})^n]$. Since $\{a_n\}_{n=0}^{\infty}$ is strictly increasing, we must have $a_0 - \frac{1}{5} = 0$, *i.e.* $\boxed{a_0 = \dfrac{1}{5}}$.

10. Jonathan is playing with his life savings. He lines up a penny, nickel, dime, quarter, and half-dollar from left to right. At each step, Jonathan takes the leftmost coin at position 1 and uniformly chooses a position $2 \le k \le 5$. He then moves the coin to position k, shifting all coins at positions 2 through k leftward. What is the expected number of steps it takes for the half-dollar to leave and subsequently return to position 5?

Solution. Let T_k denote the expected number of steps required for a coin at position k to reach position 1. Note that, at each step, the probability of the coin shifting from position k to $k-1$ is $\frac{6-k}{4}$, so we expect that it will take $\frac{4}{6-k}$ steps to reach position $k-1$. This gives the recursive formula $T_k = \frac{4}{6-k} + T_{k-1}$ with $T_1 = 0$, so we can compute the following:

k	1	2	3	4	5
T_k	0	1	$\frac{7}{3}$	$\frac{13}{3}$	$\frac{25}{3}$

The half-dollar will then take $\frac{25}{3}$ steps in expectation to reach position 1. Afterwards, there is a $\frac{1}{4}$ chance that the next step will return the half-dollar to position 5 and a $\frac{3}{4}$ chance that it will be taken elsewhere. This means that it takes $\frac{4}{1} - 1 = 3$ failed trials in expectation for the half-dollar to successfully return to position 5. Each failed trial is expected to take $1 + \frac{1}{3}(T_2 + T_3 + T_4)$ steps, and the successful trial takes only 1 step to return to position 5. Thus, the overall expected value is $T_5 + 3(1 + \frac{1}{3}(T_2 + T_3 + T_4)) + 1 = \boxed{20}$ steps.

4 Relay Round

Problem 1

Problem 1.1: A robot is located at 2 on the number line, and it needs to reach either 5 or 0. Every second, there's a $\frac{1}{3}$ chance it breaks down, a $\frac{1}{3}$ chance it moves one unit in the positive direction, and a $\frac{1}{3}$ chance it moves one unit in the negative direction. The probability the robot manages to reach 5 or 0 before breaking down is $\frac{m}{n}$, where m and n are coprime. Find n.

Solution. Let P_n for $1 \le n \le 4$ be the probability that starting at point n, the robot will manage to reach 5 or 0 before breaking down. Then we have the system

$$\begin{aligned}
P_1 &= \frac{1}{3} + \frac{1}{3}P_2,\\
P_2 &= \frac{1}{3}P_1 + \frac{1}{3}P_3,\\
P_3 &= \frac{1}{3}P_2 + \frac{1}{3}P_4,\\
P_4 &= \frac{1}{3}P_3 + \frac{1}{3},
\end{aligned}$$

solving which gives $P_2 = \frac{1}{5}$. Hence, $\boxed{n = 5}$.

Problem 1.2: Let T = TNYWR. Navya, the fruit ninja, has a bitter feud with watermelon and strawberries. She can only cut 3 watermelon with one slice or T strawberries with one slice. Suppose she slices 17 times tomorrow, and let N be the total number of watermelon and strawberries she cuts tomorrow. How many possible values of N are prime?

Solution. Suppose Navya slices strawberries s times. Then, she cuts

$$3(17 - s) + 5s = 51 + 2s.$$

Since s ranges from 0 to 17 inclusive, the possible values of N are

$$51, 53, 55, \dots, 83, 85.$$

Searching through this list, we see that the primes are

$$53, 59, 61, 67, 71, 73, 79, 83,$$

for a total of $\boxed{8}$ primes.

Problem 1.3: Let T = TNYWR and $f(x) = x^5 + 18x^4 + 19x^3 + 20x^2 + 21x + T$. The roots of f are a, b, c, d and e. Find $(a-1)(b-1)(c-1)(d-1)(e-1)$.

Solution. Notice that $f(x) = (x-a)(x-b)(x-c)(x-d)(x-e)$. Hence, we have $(a-1)(b-1)(c-1)(d-1)(e-1) = -f(1) = \boxed{-87}$.

Problem 2

Problem 2.1: $x, y \in \mathbb{R}$ satisfies $x\sqrt{y-1} + y\sqrt{x-1} = xy$. Find x.

Solution. From the problem, we know that $x, y > 1$. Hence, we suppose $x = \sec^2\alpha, y = \csc^2\beta$, where $0 < \alpha, \beta < \frac{\pi}{2}$. This gives

$$\sec^2\alpha\sqrt{\csc^2\beta - 1} + \csc^2\beta\sqrt{\sec^2\alpha - 1} = \sec^2\alpha\csc^2\beta,$$

which simplifies to $\sin 2\alpha + \sin 2\beta = 2$. Hence, we have $\sin 2\alpha = \sin 2\beta = 1$, which gives $\alpha = \beta = \frac{\pi}{4}$. Thus, we have $\boxed{x = 2}$.

Problem 2.2: Let T = TNYWR. A sequence $\{a_n\}$ satisfies that for any $m, n \in \mathbb{N}$ such that $m \geq n$ we have $a_{m+n} + a_{m-n} = \frac{1}{T}(a_{2m} + a_{2n})$. Given $a_1 = 1$, find the last digit of a_{2023}.

Solution. We observe that

$$\frac{1}{2}(a_{2m} + a_{2m}) = a_{2m} + a_0 = 2(a_m + a_m),$$

which gives $a_0 = 0$ and $a_{2m} = 4a_m$. Then, we can easily show that $a_m = m^2$ with induction. Hence, the last digit of a_{2023} is $\boxed{9}$.

Problem 2.3: Let T = TNYWR. The sequence $\{a_n\}$ satisfies $a_1 = 7$ and the recurrence relation

$$a_{n+1} = Ta_n + 7.$$

Find the sum of all values of i such that a_i is a divisor of a_{88}.

Solution. We write the numbers in base 9. Then note that

$$a_1 = (7)_9, \quad a_2 = (77)_9, \quad a_3 = (777)_9, \quad \cdots$$

This pattern holds because multiplying a_n by 9 moves the decimal place over when in base 9. So, in general, a_n is n sevens in base 9. Now we consider when $a_i \mid a_m$ for positive integers $i \leq m$. To do so, note that if we use long division, we get that

$$a_m \equiv a_{m \pmod{i}} \pmod{a_i},$$

where we let $a_0 = 0$ for notational purposes. From this, it's clear that $a_i \mid a_m$ iff $i|m$. So, our answer is the sum of the divisors of $88 = 2^3 \cdot 11$, which is

$$(1 + 2 + 2^2 + 2^3)(1 + 11) = (15)(12) = \boxed{180}.$$

5 Devil Round

1. Number of undergraduate students at Duke in fall 2021.

   Answer: $\boxed{6789}$

2. Number of graduate and professional students at Duke in fall 2021.

   Answer: $\boxed{9991}$

3. Maximum number of people that Duke chapel can hold.

   Answer: $\boxed{1800}$

4. Height of Duke chapel (in feet).

   Answer: $\boxed{210}$

5. Number of computer stations in Bostock library.

   Answer: $\boxed{96}$

6. The maximum prime factor of the year in which Duke was created.

   Answer: $\boxed{37}$

7. Number of tenured professors at Duke in fall 2021.

   Answer: $\boxed{1650}$

8. Median age of Duke alumni by summer 2021.

   Answer: $\boxed{47}$

9. Percentage of undergraduate receiving financial aid.

   Answer: $\boxed{52}$

10. Percentage of undergraduates that successfully graduate in four years.

    Answer: $\boxed{95}$

6 Tiebreaker

Problem 1: The sequence $\{x_n\}$ is defined by

$$x_{n+1} = \begin{cases} 2x_n - 1, & \text{if } \frac{1}{2} \leq x_n < 1 \\ 2x_n, & \text{if } 0 \leq x_n < \frac{1}{2} \end{cases}$$

where $0 \leq x_0 < 1$ and $x_7 = x_0$. Find the number of sequences satisfying these conditions.

Solution. First, we observe that the sequence $\{x_n\}$ is fully determined once x_0 is determined. Hence, we essentially need to find the number of different choices for x_0.

We represent the numbers in binary. Suppose $x_n = (0.b_1b_2\cdots)_2$. If $b_1 = 1$, then we have $\frac{1}{2} \le x_n < 1$ and $x_{n+1} = 2x_n - 1 = (0.b_2b_3\cdots)_2$. If $b_1 = 0$, then we have $0 \le x_n < \frac{1}{2}$ and $x_{n+1} = 2x_n = (0.b_2b_3\cdots)_2$. Hence, given $x_n = (0.b_1b_2\cdots)_2$, we always have $x_{n+1} = (0.b_2b_3\cdots)_2$.

Suppose $x_0 = (0.a_1a_2\cdots)_2$. Then, we have $x_7 = (0.a_8a_9\cdots)_2$. Since $x_0 = x_7$, we have $a_i = a_{i+7}$ for all $i \in \mathbb{N}^*$. Hence, we only need to count the number of choices for $\{a_1, \ldots, a_7\}$. Since we have 2 choices for each of them, we have $2^7 - 1 = \boxed{127}$ choices in total.

Problem 2: Let $M = \{1, \ldots, 2022\}$. For any nonempty set $X \subseteq M$, let a_X be the sum of the maximum and the minimum number of X. Find the average value of a_X across all nonempty subsets X of M.

Solution. For any nonempty subset X of M, let $X' = \{2023 - x \mid x \in X\}$. Then, X' is a nonempty subset of M as well. Moreover, if $X \ne Y$, then $X' \ne Y'$. Hence, we can pair up X and X', so that $a_X + a_{X'} = 2 \cdot 2023 = 4046$ by definition of X'. The only case in which we can not do such pairing is when $X = X'$. However, we must have $a_X = 2023$ in this case. Hence, the average value of a_X across all nonempty subsets X of M is $\boxed{2023}$.