# Team Round

## DMM 2022

1. The serpent of fire and the serpent of ice play a game. Since the serpent of ice loves the lucky number 6, he will roll a fair 6-sided die with faces numbered 1 through 6. The serpent of fire will pay him $\log_{10} x$, where $x$ is the number he rolls. The serpent of ice rolls the die 6 times. His expected total amount of winnings across the 6 rounds is $k$. Find $10^k$.

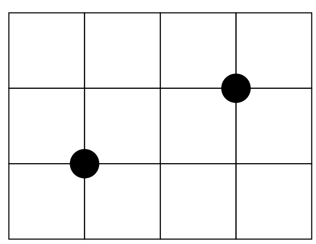
2. Let $a = \log_3 5$, $b = \log_3 4$, $c = -\log_3 20$, evaluate $\frac{a^2+b^2}{a^2+b^2+ab} + \frac{b^2+c^2}{b^2+c^2+bc} + \frac{c^2+a^2}{c^2+a^2+ca}$.

3. Let $\triangle ABC$ be an isosceles obtuse triangle with $AB = AC$ and circumcenter $O$. The circle with diameter $AO$ meets $BC$ at points $X, Y$, where $X$ is closer to $B$. Suppose $XB = YC = 4$, $XY = 6$, and the area of $\triangle ABC$ is $m\sqrt{n}$ for positive integers $m$ and $n$, where $n$ does not contain any square factors. Find $m + n$.

4. Alice is not sure what to have for dinner, so she uses a fair 6-sided die to decide. She keeps rolling, and if she gets all the even numbers (*i.e.* getting all of 2, 4, 6) before getting any odd number, she will reward herself with McDonald's. Find the probability that Alice could have McDonald's for dinner.

5. How many distinct ways are there to split 50 apples, 50 oranges, 50 bananas into two boxes, such that the products of the number of apples, oranges, and bananas in each box are non-zero and equal?

6. Sujay and Rishabh are taking turns marking lattice points within a square board in the Cartesian plane with opposite vertices $(1, 1), (n, n)$ for some constant $n$. Sujay loses when the two-point pattern $P$ below shows up:



   That is, Sujay loses when there exists a pair of points $(x, y)$ and $(x + 2, y + 1)$. He and Rishabh stop marking points when the pattern $P$ appears on the board. If Rishabh goes first, let $S$ be the set of all integers $3 \leq n \leq 100$ such that Rishabh has a strategy to always trick Sujay into being the one who creates $P$. Find the sum of all elements of $S$.

7. Let $a$ be the shortest distance between the origin $(0, 0)$ and the graph of $y^3 = x(6y - x^2) - 8$. Find $\lfloor a^2 \rfloor$. ($\lfloor x \rfloor$ is the largest integer not exceeding $x$)

8. Find all real solutions to the following equation:

$$2\sqrt{2}x^2 + x - \sqrt{1 - x^2} - \sqrt{2} = 0.$$

9. Given the expression $S = (x^4 - x)(x^2 - x^3)$ for $x = \cos\frac{2\pi}{5} + i\sin\frac{2\pi}{5}$, find the value of $S^2$.

10. In a 32 team single-elimination rock-paper-scissors tournament, the teams are numbered from 1 to 32. Each team is guaranteed (through incredible rock-paper-scissors skill) to win any match against a team with a higher number than it, and therefore will lose to any team with a lower number. Each round, teams who have not lost yet are randomly paired with other teams, and the losers of each match are eliminated. After the 5 rounds of the tournament, the team that won all 5 rounds is ranked 1st, the team that lost the 5th round is ranked 2nd, and the two teams that lost the 4th round play each other for 3rd and 4th place. What is the probability that the teams numbered 1, 2, 3, and 4 are ranked 1st, 2nd, 3rd, and 4th respectively? If the probability is $\frac{m}{n}$ for relatively prime integers $m$ and $n$, find $m$.