

7.

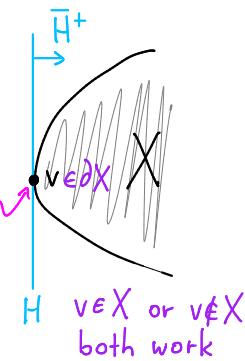
Def:  $H$  is a support hyperplane of  $X$  at  $v \in \overline{X}$  if  $v \in H$  and  $X \subseteq \overline{H}^+$  or  $\overline{H}^-$

compare: separating has  $X \subseteq \overset{\circ}{H}^+$ , so supporting is "closer" to  $X$

Thm:  $X$  convex and  $v \in \partial X \Rightarrow \exists$  support  $H$  of  $X$  at  $v$ .

think  $v \in \partial X$ ,  
though not  
necessary by def

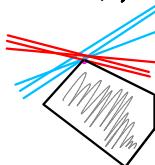
Pf:  $X \subseteq \overline{H}^+ \Rightarrow \overline{X} \subseteq \overline{H}^+$  so assume  $X = \overline{X}$ . Fix a norm  $\|\cdot\|$  on  $V^*$ . extreme point



For  $k \in \mathbb{N}$  pick  $v_k \notin X$  and  $l_k \in S^{n-1} = \{w \in V^* \mid \|w\|=1\} \subseteq V^*$  separating  $v_k$  from  $X$ .

Assume  $v_k \rightarrow v$  as  $k \rightarrow \infty$ .  $S^{n-1}$  is compact (closed + bounded) so

$\{l_k\}$  has a convergent subsequence; replace  $\{l_k\}$  with that to get  $\{l_k\} \rightarrow l$ .



$$\forall x \in X, \underbrace{\{l_k(x) - l_k(v_k)\}_{k \in \mathbb{N}}}_{\geq 0} \rightarrow \underbrace{l(x) - l(v)}_{\geq 0} \Rightarrow H = \{y \in V \mid l(y) = l(v)\} \text{ suffices. } \square$$

If you weren't convinced  
that  $V^*$  should be  
thought about separately  
from  $V$ , then  
reconsider now.

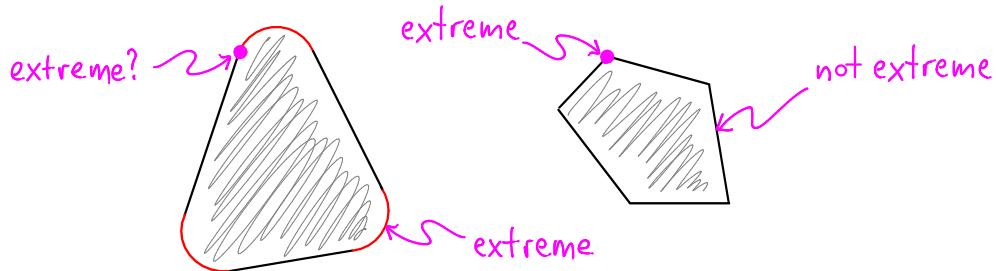
Def:  $S \subseteq V$  has convex hull  $\text{conv}(S) = \{\text{convex combinations of points in } S\}$

Prop: = smallest convex set  $\supseteq S$ .

Pf: A convex combination of convex combinations is a convex combination. Now use Property 8.  $\square$

Def: For  $X$  convex,  $v \in X$  is an extreme point if  $v \notin \text{conv}(X \setminus \{v\})$ .

Lemma:  $\Leftrightarrow v \notin \overline{xy}$  whenever  $v \neq x \in X$  and  $v \neq y \in X$ . i.e. two points suffice to witness  $v$



Thm:  $X = \overline{X}$  bounded + convex  $\Rightarrow$  each supporting  $H$  contains an extreme point of  $X$ .

Pf (assuming  $\dim V < \infty$ ): Fix supporting  $H$ . Set  $Y = X \cap H$ : bounded, closed, convex.

Claim:  $v \in Y$  extreme in  $Y \Rightarrow v$  extreme in  $X$ .  $\dim V = 1$  elementary

$\Rightarrow$  Thm by induction on  $\dim V$  (as  $\dim H = \dim V - 1$ ) via previous Thm.

All that's really  
needed for Thm  
is for  $Y$  to contain  
any extreme point of  $X$ .

Pf of Claim: Pick  $l \perp H$  with  $l(X) \geq l(v)$ .

$$v = \alpha x + \beta y \text{ with } x, y \in X \text{ and } \alpha + \beta = 1 \Rightarrow l(v) = \alpha l(x) + \beta l(y)$$

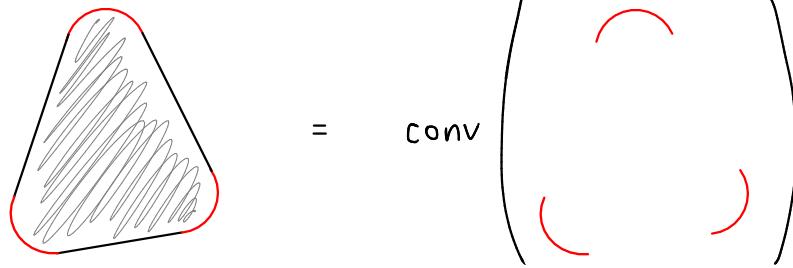
no inner product  
is being used here

$$\Rightarrow l(x) = l(v) = l(y) \text{ since } l(x) \geq l(v) \text{ and } l(y) \geq l(v)$$

$$\Rightarrow x \in H \text{ and } y \in H. \quad \square$$

Thm (Krein–Milman):  $X = \overline{X}$  bounded + convex  $\Rightarrow X = \text{conv}(\text{extreme points of } X)$ .

E.g.



Pf: Suppose  $X \supseteq Y$  with  $Y = \overline{Y}$  convex. think:  $Y = \text{conv}(\text{extreme points of } X)$

$X \not\subseteq Y \Rightarrow \exists v \in X \setminus Y$  with  $l$  separating  $v$  from  $Y$ .

$X$  closed + bounded (i.e. compact)

$\Rightarrow l$  attains min. at some  $x \in X$ ;

note that  $l(x) < l(Y)$  by separation.

$H = \{w \in V \mid l(w) = l(x)\}$  supports  $X$  at  $x$  by construction.

Previous thm  $\Rightarrow \exists$  extreme point of  $X$  in  $H$

$\Rightarrow Y$  can't contain all extreme points of  $X$ .

Thus  $X \subseteq \text{conv}(\text{extreme points of } X)$ .

$\supseteq$  by Prop.  $\square$

