

11.

Unitary matrices $\mathbb{C}^n \cong \mathbb{R}^{2n}$  via  $(a_1+b_1i, \dots, a_n+b_ni)$ double the size  
of each entry

$$\begin{array}{c} \parallel \\ M_n \mathbb{C} \end{array}$$

$$\begin{array}{c} \parallel \\ M_{2n} \mathbb{R} \end{array}$$

$$(a_1, b_1, \downarrow, \dots, a_n, b_n)$$

induces  $d_n: \mathbb{C}^{n \times n} \rightarrow \mathbb{R}^{2n \times 2n}$  by viewing each  $\mathbb{C}$ -entry as a  $2 \times 2$  block

$$a+bi \mapsto \begin{bmatrix} a & -b \\ b & a \end{bmatrix} \quad \text{or} \quad 1 \mapsto \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$\begin{array}{c} \parallel \\ M_n \mathbb{C} \end{array}$$

Def:  $A \in M_{2n} \mathbb{R}$  is complex-linear if  $A \in \text{image}(d_n)$ .Prop:  $A \in U_n \Leftrightarrow d_n(A) \in O_{2n}(\mathbb{R}) \cap d_n(M_n \mathbb{C})$ .

$$\text{Pf: } d_n(A^*) = d_n(A)^*$$

adjoint      ordinary transpose

$$-i \mapsto \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}^T$$

Check:  $d_n(AB) = d_n(A)d_n(B)$ .  $d_n$  is a ring homomorphism.

- view  $A, B$  as  $\mathbb{C}$ -linear on  $\mathbb{C}^n$
- compose
- view  $\mathbb{C}^n$  as v.s./ $\mathbb{R}$

- view  $A, B$  as operators on  $\mathbb{R}^{2n}$
- compose

The two sides are equal as  
functions  $\mathbb{C}^n \rightarrow \mathbb{C}^n$ , regardless  
of whether  $\mathbb{C}^n$  is viewed as  
v.s./ $\mathbb{C}$  or v.s./ $\mathbb{R}$  or merely as a set.

$$\text{Thus } d_n(A)d_n(A)^* = d_n(A)d_n(A^*) = d_n(AA^*)$$

$$d_n(A) \in O_{2n}(\mathbb{R}) \Leftrightarrow \begin{array}{c} \parallel \\ I_{2n} \end{array}$$

$$\begin{array}{c} \parallel \\ I_{2n} \end{array} \mathbb{R} \Leftrightarrow AA^* = I_n^{\mathbb{C}} \Leftrightarrow A \in U_n. \quad \square$$

Def: For v.s.  $V/\mathbb{F}$  with  $\langle \cdot, \cdot \rangle$ ,  $O(V) = \{ \varphi: V \rightarrow V \mid \langle \varphi_x, \varphi_y \rangle = \langle x, y \rangle \forall x, y \in V \}$ .Prop:  $O(V) = \{ \varphi \in GL(V) \mid \|\varphi_x\| = \|x\| \forall x \in V \}$ .  $\Rightarrow O(\mathbb{F}^n) = O_n(\mathbb{F})$ 

$$\text{Pf: } \|x-y\|^2 = \langle x-y, x-y \rangle_{\mathbb{R}} = \|x\|^2 + \|y\|^2 - 2\langle x, y \rangle_{\mathbb{R}} \quad \langle x, y \rangle_{\mathbb{C}} = \langle d_n x, d_n y \rangle_{\mathbb{R}} + i \langle d_n x, d_n(iy) \rangle_{\mathbb{R}}$$

$$\Rightarrow \langle x, y \rangle_{\mathbb{R}} = \frac{1}{2} (\|x\|^2 + \|y\|^2 - \|x-y\|^2)$$

so  $\varphi$  preserves norm  $\Leftrightarrow$  preserves inner product  $_{\mathbb{R}}$ .For  $\mathbb{F} = \mathbb{C}$ ,  $\varphi: \mathbb{C}^n \rightarrow \mathbb{C}^n$  preserves norms  $\Rightarrow d_n(\varphi)$  preserves norms check!For general  $V/\mathbb{C}$ , choose orthonormal basis

$$\Rightarrow d_n(\varphi) \in O(\mathbb{R}^{2n})$$

$$(V, \langle \cdot, \cdot \rangle) \cong (\mathbb{C}^n, \|\cdot\|_2)$$

$$\Rightarrow \varphi \in O(\mathbb{C}^n). \quad \square$$

Prop:  $A \in O_n(\mathbb{F}) \Rightarrow |\det A| = 1$ .

$$\text{Pf: } \det(AA^*) = (\det A)(\overline{\det A}) = |\det A|^2 = 1 \text{ if } AA^* = I. \quad \square$$

Def:  $SO_n(\mathbb{R}) = \{ A \in O_n(\mathbb{R}) \mid \det A = 1 \}$ . could have been  $\pm 1$ 

$$SU_n = \{ A \in U_n \mid \det A = 1 \} \quad " \quad " \quad " \quad e^{i\theta} \text{ for some } \theta \in \mathbb{R}$$

Running assumption:  $V$  with  $\langle \cdot, \cdot \rangle$ ,  $\dim_{\mathbb{C}} V = n$ ,  $\varphi: V \rightarrow V$   $\mathbb{C}$ -linear

Thm:  $V$  has orthonormal basis  $B$  with  $[\varphi]_B$  upper- $\Delta$ .

$\Leftrightarrow A \in M_n \mathbb{C} \Rightarrow U^*AU$  upper- $\Delta$  for some  $U \in U_n$ .

Pf:  $n=1$  ✓

$n \geq 2$ : pick unit eigenvector  $u_1$  and  $\perp$  normal basis  $v_2, \dots, v_n$  for  $u_1^\perp$  with  $Q^*AQ = \begin{bmatrix} \lambda_1 & * \\ 0 & \boxed{A'} \end{bmatrix}$

where  $Q = \begin{bmatrix} 1 & 1 & 1 \\ u_1 & v_2 & \cdots & v_n \\ 1 & 1 & 1 \end{bmatrix}$ . By induction, choose  $W = \begin{bmatrix} 1 & 1 \\ w_2 & \cdots & w_n \\ 1 & 1 \end{bmatrix} \in U_{n-1}$ , so  $W^*A'W$  upper- $\Delta$ .

Set  $U = \begin{bmatrix} 1 \\ W \end{bmatrix} Q$ . Then  $U^*AU = \begin{bmatrix} 1 \\ W^* \end{bmatrix} Q^*AQ \begin{bmatrix} 1 \\ W \end{bmatrix} = \begin{bmatrix} 1 \\ W^* \end{bmatrix} \begin{bmatrix} 1 & * \\ \boxed{A'} & \boxed{W} \end{bmatrix} = \begin{bmatrix} 1 & * \\ \boxed{*} & \boxed{*} \end{bmatrix}$ . □

Note: Thm + pf work  $\mathbb{R}$  if all eigenvalues assumed  $\in \mathbb{R}$ .

Cor (Spectral thm):  $\varphi = \varphi^* \Rightarrow V$  has orthonormal basis of eigenvectors and all eigenvalues  $\in \mathbb{R}$ .

$\Leftrightarrow A = A^* \Rightarrow A = UDU^*$  for some unitary  $U$  and real diagonal  $D$ .

Pf: Thm  $\Rightarrow A = UBU^*$  with  $B$  upper- $\Delta$ . But  $\xrightarrow{\text{unitarily similar to real diagonal}}$   
 $\xrightarrow{\parallel} A^* = U B^* U^* \Rightarrow B = B^* \Rightarrow B$  diagonal and real. □

Def:  $\varphi$  is normal if  $\varphi\varphi^* = \varphi^*\varphi$ .  $\varphi$  commutes with its adjoint. E.g.  $\varphi \in U_n$

Cor 2:  $\varphi$  normal  $\Rightarrow V$  has orthonormal basis of eigenvectors.

$\Leftrightarrow A$  normal  $\Rightarrow A = UDU^*$  for some  $U \in U_n$  and diagonal  $D$ .  $\xrightarrow{\text{real}}$

Pf: Suffices by Thm: normal upper- $\Delta$  is diagonal. Same induction as for Thm.

Assume  $N$  normal upper- $\Delta$ . Check

- $(N^*N)_{11} = \overline{a_{11}} a_{11} = |a_{11}|^2$
- $(N N^*)_{11} = |a_{11}|^2 + \underbrace{|a_{12}|^2 + \cdots + |a_{1n}|^2}_{=0}$

$$\Rightarrow = 0$$

$$\Rightarrow \overline{a_{12}} \quad \overline{a_{1n}}$$

$\Rightarrow N^*N$  and  $NN^*$  computed block by block

$\Rightarrow$  upper- $\Delta$  is normal

$\Rightarrow$  " " diagonal by induction. □

$$N = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ 0 & \boxed{\quad} & * & \cdots \\ \vdots & \ddots & \ddots & \ddots \\ 0 & \cdots & \cdots & \ddots \end{bmatrix}$$

upper- $\Delta$

Prop:  $\varphi$  normal  $\Leftrightarrow \|\varphi_x\| = \|\varphi_x^*\| \quad \forall x \in V$ .

Pf:  $\Rightarrow: \|\varphi_x\|^2 = \langle \varphi_x, \varphi_x \rangle = \langle \varphi^* \varphi_x, x \rangle = \langle \varphi \varphi_x^*, x \rangle = \langle \varphi_x^*, \varphi_x^* \rangle = \|\varphi_x^*\|^2$ .

omitted  $\Leftarrow:$  Compare  $\langle \varphi^* \varphi_x, y \rangle = \langle \varphi_x, \varphi_y \rangle$  to  $\langle \varphi \varphi_x^*, y \rangle = \langle \varphi_x^*, \varphi_y^* \rangle$  by expressing  $\langle \cdot, \cdot \rangle$  in terms of  $\|\cdot\|$ . □