

15.

Def:  $d(\tilde{\lambda}, \Lambda) = \text{distance from } \tilde{\lambda} \in \mathbb{C} \text{ to closed (here, finite) } \Lambda : \min_{\lambda \in \Lambda} |\tilde{\lambda} - \lambda|$ .

Fix  $A \in \mathbb{C}^{n \times n}$  and  $\tilde{A} = A + E$ .

1. spectral variation  $sv_A(\tilde{A}) = \max_{\tilde{\lambda} \in \Lambda(\tilde{A})} d(\tilde{\lambda}, \Lambda(A))$  each  $\tilde{\lambda}$  has a closest  $\lambda(\tilde{\lambda}) \in \Lambda(A)$ ;  $sv_A(\tilde{A}) = \max_{\tilde{\lambda}} |\tilde{\lambda} - \lambda(\tilde{\lambda})|$

$\text{not symmetric} \Rightarrow \text{not a metric: } n=2, \lambda_1 = \tilde{\lambda}_1 = \tilde{\lambda}_2 = 0 \text{ and } \lambda_2 = 1 \Rightarrow sv_A(\tilde{A}) = 0 \text{ but } A \neq \tilde{A}$ .

↑ geometric interpretation:  $\Lambda(\tilde{A}) \subseteq \bigcup_{i=1}^n D_i$  where  $D_i = \{z \in \mathbb{C} \mid |z - \lambda_i| \leq sv_A(\tilde{A})\}$   
so symmetrize:

2. Hausdorff distance  $hd(A, \tilde{A}) = \max \{sv_A(\tilde{A}), sv_{\tilde{A}}(A)\}$ .

How much must  $\Lambda$  be fattened to swallow  $\tilde{\Lambda}$ ?

3. matching distance  $md(A, \tilde{A}) = \min_{\text{permutations } \pi \in S_n} \{ \max_i |\tilde{\lambda}_{\pi(i)} - \lambda_i| \} = \min \text{length}(\text{longest edge in perfect matching})$

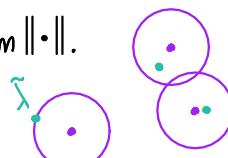
Lemma (Hadamard's inequality):  $|\det A| \leq \prod_{i=1}^n \|\text{col}_i \text{ of } A\|_2$ .  $\text{vol}(\text{parallelepiped}) \leq \prod_i \|\text{edges}\|_2$

Pf: True for  $A$  upper- $\Delta$ , and both sides unchanged by  $A \mapsto UA$  for unitary  $U$ .

Use Schur decomposition.  $\square$

Elsner's Thm:  $hd(A, \tilde{A}) \leq (\|A\| + \|\tilde{A}\|)^{1-\frac{1}{n}} \|E\|^{\frac{1}{n}} = \beta$  for operator norm  $\|\cdot\|$ .

Pf:  $\beta$  symmetric in  $A$  and  $\tilde{A}$ , so need only  $sv_A(\tilde{A}) \leq \beta$ .



Suppose  $sv_A(\tilde{A}) = d(\tilde{\lambda}, \Lambda)$ . So  $\tilde{\lambda}$  is the last eigenvalue of  $\tilde{A}$  to be swallowed.

Pick  $\perp$  normal basis  $x_1, \dots, x_n$  with  $\tilde{A}x_i = \tilde{\lambda}x_i$ . Then

$$sv_A(\tilde{A})^n \leq \prod_{\lambda \in \Lambda} |\tilde{\lambda} - \lambda| \quad d(\tilde{\lambda}, \Lambda) \leq |\tilde{\lambda} - \lambda| \quad \forall \lambda \in \Lambda$$

$$= |\det(A - \tilde{\lambda}I)| \quad \det = \prod \text{eigenvalues and } \lambda \in \Lambda(A) \Leftrightarrow \tilde{\lambda} - \lambda \in \Lambda(A - \tilde{\lambda}I)$$

$$\leq \prod_{i=1}^n \|(A - \tilde{\lambda}I)x_i\|_2 \quad \text{by Lemma}$$

$$(A - \tilde{\lambda}I)x_i = \|(A - \tilde{\lambda}I)x_i\|_2 \prod_{i=2}^n \|(A - \tilde{\lambda}I)x_i\|_2$$

$$\leq \|Ex_1\|_2 \prod_{i=2}^n (\|Ax_i\|_2 + \|\tilde{A}x_i\|_2)$$

$$\leq \|E\| \left( \|A\| + \|\tilde{A}\| \right)^{n-1}. \quad \text{Take } n^{\text{th}} \text{ root. } \square$$

$$\sqrt{\max \Lambda(\tilde{A}^T \tilde{A})}$$

$$\text{E.g. } (***) \begin{bmatrix} 1 & 10^{-4} \\ 10^{-4} & 2 \end{bmatrix} = \begin{bmatrix} 1 & \\ & 2 \end{bmatrix} + \begin{bmatrix} & 10^{-4} \\ 10^{-4} & \end{bmatrix} \Rightarrow hd(A, \tilde{A}) \leq (2 + 2 + 10^{-8})^{1/2} (10^{-4})^{1/2} < (2 + 10^{-7}) \cdot 10^{-2} < .021$$

$$P_{\tilde{A}}(z) = z^2 - 3z + 2 - 10^{-8}$$

$$\Rightarrow \tilde{\Lambda} \subseteq [0.979, 1.021] \cup [1.979, 2.021]$$

$$\frac{3}{2} \pm \frac{1}{2} \sqrt{1 + 4 \cdot 10^{-8}} = \frac{3}{2} \pm \frac{1}{2} \pm 10^{-8} + O(10^{-16})$$

$$\{1 - \varepsilon, 2 + \varepsilon\} \text{ for } \varepsilon = 10^{-8} + O(10^{-16})$$

→ pretty bad bound

Thm (Ostrowski, Elsner):  $md(A, \tilde{A}) \leq n\beta$ . Pf omitted.

Bauer-Fike Thm:  $\|\cdot\|$  consistent on  $\mathbb{C}^{n \times n}$  and  $\tilde{\lambda} \in \tilde{\Lambda} \setminus \Lambda \Rightarrow \|(\underbrace{A - \tilde{\lambda}I}_{\text{singular}})^{-1}\|^{-1} \leq \|E\|$ .

used in Pf of Gershgorin (32)

$$\begin{aligned} \text{Pf: } \underbrace{\tilde{A} - \tilde{\lambda}I}_{\text{singular}} &= A - \tilde{\lambda}I + E = (\underbrace{A - \tilde{\lambda}I}_{\text{invertible}})(\underbrace{I + (A - \tilde{\lambda}I)^{-1}E}_{\text{big}}) \\ &\Rightarrow 1 \leq \|(\tilde{A} - \tilde{\lambda}I)^{-1}E\| \quad (*) \\ &\leq \|(\tilde{A} - \tilde{\lambda}I)^{-1}\| \|E\|. \quad \square \end{aligned}$$

" $\tilde{\lambda}$  is close to some  $\lambda \in \Lambda$ , bounded by norm of  $E$ ."

E.g.  $A = \text{diag}(\lambda_1, \dots, \lambda_n) \Rightarrow (A - \tilde{\lambda}I)^{-1}$  has "max entry"  $(\lambda - \tilde{\lambda})^{-1}$ , so thm  $\Rightarrow \lambda - \tilde{\lambda}$  can't be too big.

Gershgorin's Thm: For  $A \in \mathbb{C}^{n \times n}$ , let  $\alpha_i = \sum_{j \neq i} |a_{ij}|$  and  $G_i(A) = B_{\alpha_i}(a_{ii}) = \{z \in \mathbb{C} \mid |z - a_{ii}| \leq \alpha_i\}$ .

Then  $\Lambda \subseteq \bigcup_{i=1}^n G_i(A)$ . Equivalently,  $\lambda \in \Lambda \Rightarrow \lambda \in G_i(A)$  for some  $i$ .

Pf: Fix  $\lambda \in \Lambda$ .  $\lambda = a_{ii} \Rightarrow \lambda \in G_i$ , so assume  $\lambda \neq a_{ii} \forall i$ .

In Bauer-Fike thm view  $A$  as perturbation of  $D = \begin{bmatrix} a_{11} & 0 \\ \vdots & \ddots & 0 \\ 0 & \cdots & a_{nn} \end{bmatrix}$ , so  $A = D + E$  ( $E = A - D$ ):

$$\begin{aligned} (*) \Rightarrow 1 &\leq \underbrace{\|(D - \lambda I)^{-1}E\|_\infty}_{\Rightarrow |a_{ii} - \lambda| \leq \sum_{j \neq i} |a_{ij}| \text{ for some } i. \quad \square} = \max_i |a_{ii} - \lambda|^{-1} \sum_{j \neq i} |a_{ij}|, \quad \text{where } \|C\|_\infty = \max_i \sum_{j=1}^n |c_{ij}|. \end{aligned}$$

Def:  $G_i(A) = i^{\text{th}}$  Gershgorin disk.

Note: More is true:  $G_{i_1} \cup \dots \cup G_{i_m}$  disjoint from the other  $n-m$

$$\Rightarrow \#\Lambda \cap (G_{i_1} \cup \dots \cup G_{i_m}) = m. \quad \text{HW4 (apply continuity)}$$

$$\text{E.g. } (***) \begin{bmatrix} 1 & 10^{-4} \\ 10^{-4} & 2 \end{bmatrix} = \begin{bmatrix} 1 & \\ & 2 \end{bmatrix} + \begin{bmatrix} & 10^{-4} \\ 10^{-4} & \end{bmatrix} \Rightarrow \tilde{\Lambda} \subseteq [.9999, 1.0001] \cup [1.9999, 2.0001]$$

better, but still 4 orders of magnitude off.

$$\text{Trick: } \tilde{A} \sim \begin{bmatrix} \gamma & \\ & 1 \end{bmatrix} \tilde{A} \begin{bmatrix} \gamma^{-1} & \\ & 1 \end{bmatrix} = \begin{bmatrix} 1 & 10^{-4}\gamma \\ 10^{-4}\gamma^{-1} & 2 \end{bmatrix}$$

has same  $\tilde{\Lambda}$  but different  $G_i(\tilde{A})$ !

Choose  $\gamma$  small but with  $10^{-4}\gamma + 10^{-4}\gamma^{-1} < 1$  so  $\tilde{G}_1 \cap \tilde{G}_2 = \emptyset$ .

Suffices:  $\gamma^{\pm 1} = 10^{-4} + 10^{-11} \Rightarrow \tilde{\Lambda} \subseteq [1-\delta, 1+\delta] \cup [2-\delta, 2+\delta]$

$$\text{for } \delta = 10^{-8} + 10^{-15} \quad \text{Quite sharp!}$$

$$\text{Check: } 10^{-4}(1+10^{-7})^{-1} = 10^{-4}(1 - 10^{-7} + 10^{-14} - \dots)$$

$$\Rightarrow 10^{-4}\gamma + 10^{-4}\gamma^{-1} = (10^{-8} + 10^{-15}) + (1 - 10^{-7} + 10^{-14} - \dots)$$

$$< 1 - 10^{-7} + 2 \cdot 10^{-8}.$$