

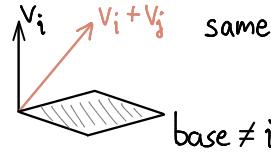
21.

Exterior algebra

Def: An alternating operator  $\varphi: \underbrace{V \times \cdots \times V}_r = V^{\times r} \rightarrow W$  is a multilinear map such that  $v_1, \dots, v_r$  linearly dependent  $\Rightarrow \varphi(v_1, \dots, v_r) = 0$ .

E.g. volume of parallelepiped on  $v_1, \dots, v_n \in \mathbb{R}^n$ :  $\text{vol} = 0$  if  $v_1, \dots, v_n$  linearly dependent

- $\xrightarrow{\text{def}} \text{vol} = \det \begin{cases} \cdot v_i \mapsto \alpha v_i \Rightarrow \text{vol} \mapsto \alpha \text{vol} \quad \forall \alpha \quad \text{including } \alpha < 0: \text{signed or oriented volume} \\ \cdot v_i \mapsto v_i + v_j \text{ for } j \neq i \Rightarrow \text{vol unchanged} \\ \cdot \text{vol}(e_1, \dots, e_n) = 1. \end{cases}$



Def:  $r^{\text{th}}$  exterior power of  $V$  is a universal alternating operator:

alternating map  $\Lambda^r: V^{\times r} \rightarrow U$  such that  $\forall \varphi$  alternating  $\exists! \tilde{\varphi}$  with  $\varphi = \tilde{\varphi} \circ \Lambda^r$ .

$$\begin{array}{c} \text{alternating } \varphi \\ \xrightarrow{\quad} \exists! \tilde{\varphi} \\ \downarrow \\ W \end{array}$$

Thm:  $\Lambda^r$  exists.  $V \otimes \cdots \otimes V$

Y

Pf: Set  $U = \Lambda^r V = V^{\otimes r} / \langle \text{span}(v_1 \otimes \cdots \otimes v_r \mid \text{two of the } v\text{'s are equal}) \rangle$  r-forms

$$V^{\times r} \rightarrow \Lambda^r V$$

$(v_1, \dots, v_r) \mapsto v_1 \wedge \cdots \wedge v_r$  multilinear because factors through  $V^{\otimes r}$

wedge

alternating because  $v_i = \sum_{j>1} \alpha_j v_j \Rightarrow v_1 \wedge \cdots \wedge v_r = \sum_{j>1} \alpha_j v_j \wedge (v_2 \wedge \cdots \wedge v_r) = 0$ , and same for  $i > 1$ .

$\varphi: V^{\times r} \rightarrow W$  multilinear  $\Rightarrow \varphi$  factors through  $V^{\otimes r}$ ...

alternating  $\Rightarrow$ ... and kills Y  $\Rightarrow$  factors through  $V^{\otimes r}/Y$ .  $\square$ 

E.g.  $v, w \in \mathbb{R}_{\text{col}}^4$  (get from class)  $\Rightarrow v \wedge w = \sum_{e_i} e_i \wedge e_j$

Prop:  $V \xrightarrow{\varphi} W$  linear induces canonical linear map  $\Lambda^r V \xrightarrow{\Lambda^r \varphi} \Lambda^r W$ .  $\Lambda^r$  is a functor.

$$v_1 \wedge \cdots \wedge v_r \mapsto \varphi(v_1) \wedge \cdots \wedge \varphi(v_r)$$

Pf: HW5, including entries of matrix if A is given.  $\square$

Quintessential E.g.:  $V = W$  and  $r = n = \dim V$ : determinant of  $\varphi: V \rightarrow V$  is  $\det \varphi = \Lambda^n \varphi$ .

Note:  $\det \varphi = \Lambda^{top} \varphi$  since  $\Lambda^r V = 0$  for  $r \geq n+1$ .

$$\begin{aligned} \varphi(e_j) = v_j = \sum_{i=1}^n a_{ij} e_i \Rightarrow \Lambda^n \varphi(e_1 \wedge \cdots \wedge e_n) &= v_1 \wedge \cdots \wedge v_n = \left( \sum_{i=1}^n a_{1i} e_i \right) \wedge \cdots \wedge \left( \sum_{i=1}^n a_{ni} e_i \right) = \sum_{\pi} e_{\pi(1)} \wedge \cdots \wedge e_{\pi(n)} \\ &= \sum_{i_1, \dots, i_n} a_{i_1 1} e_{i_1} \wedge \cdots \wedge a_{i_n n} e_{i_n}. \end{aligned}$$

Terms are 0 unless  $i_1, \dots, i_n$  distinct, so  $i_j = \pi(j)$  for some permutation  $\pi \in S_n$ . Thus

$$\begin{aligned}
 v_1 \wedge \cdots \wedge v_n &= \sum_{\pi \in S_n} a_{\pi(1)1} e_{\pi(1)} \wedge \cdots \wedge a_{\pi(n)n} e_{\pi(n)} \\
 &= \underbrace{\sum_{\pi \in S_n} (-1)^{\text{inv } \pi} a_{\pi(1)1} \cdots a_{\pi(n)n}}_{\det A} e_1 \wedge \cdots \wedge e_n
 \end{aligned}$$

Notation: For  $\sigma = \{s_1 < \cdots < s_r\} \subseteq [n]$  and  $E = e_1, \dots, e_n \in V$  set

$$e_\sigma = e_{s_1} \wedge \cdots \wedge e_{s_r}$$

$$\text{and } \Lambda^r E = \{e_\sigma \mid \sigma \in \binom{[n]}{r}\}.$$

Prop:  $E$  is a basis for  $V \Rightarrow \Lambda^r E$  spans  $\Lambda^r V$ .

Pf:  $\Lambda^r E = \{\text{squarefree elements of basis } E^{\otimes r} \text{ of } V^{\otimes r}\}$

and nonsquarefree elements  $\mapsto 0$  in  $\Lambda^r V$ .  $\square$

Q. coeff. on  $e_\sigma$  in  $v_1 \wedge \cdots \wedge v_r = ?$  i.e. How can  $v_1 \wedge \cdots \wedge v_n$  be expressed as a linear combination of elements  $e_\sigma$ ?

A.  $\det A_\sigma$ , where  $A = \begin{bmatrix} | & | \\ v_1 & \cdots & v_r \\ | & | \end{bmatrix}$  and  $A_\sigma$  takes rows indexed by  $\sigma$ .

Pf: In rows from  $\sigma$ , get one  $e_i$  from each column  $j$  with coeff  $a_{ij}$ :

$$\begin{aligned}
 \sigma = \left\{ \begin{array}{l} s_1 = s_{\pi(1)} \\ s_2 = s_{\pi(2)} \\ s_3 = s_{\pi(3)} \end{array} \right\} &\rightarrow \left| \begin{array}{c} | \\ | \\ | \end{array} \right| \quad v_1 \wedge \cdots \wedge v_r = \sum_{\sigma \in \binom{[n]}{r}} \sum_{\pi \in S_r} a_{\sigma_{\pi(1)}1} e_{\sigma_{\pi(1)}} \wedge \cdots \wedge a_{\sigma_{\pi(r)}r} e_{\sigma_{\pi(r)}} \\
 &= \sum_{\sigma \in \binom{[n]}{r}} \sum_{\pi \in S_r} a_{\sigma_{\pi(1)}1} \cdots a_{\sigma_{\pi(r)}r} e_{\sigma_{\pi(1)}} \wedge \cdots \wedge e_{\sigma_{\pi(r)}} \\
 &= \sum_{\sigma \in \binom{[n]}{r}} \sum_{\pi \in S_r} a_{\sigma_{\pi(1)}1} \cdots a_{\sigma_{\pi(r)}r} \frac{(-1)^{\text{inv } \pi}}{\det A_\sigma} e_\sigma. \quad \square
 \end{aligned}$$

Thm:  $E$  is a basis for  $V \Rightarrow \Lambda^r E$  is a basis for  $\Lambda^r V$ .

Pf: spans by Prop.

independent: existence of determinants  $\Rightarrow (v_1, \dots, v_r) \mapsto \det([v_1 \cdots v_r]_\sigma)$  is alternating, so

induces  $e_\sigma^* : \Lambda^r V \rightarrow F$  with  $e_\sigma^*(e_\tau) = \delta_{\sigma,\tau}$ ; Lemma  $\Rightarrow$  independent.  $\square$

Cor:  $\dim V = n \Rightarrow \dim \Lambda^r V = \binom{n}{r}$  if  $r \leq n$  and  $\Lambda^r V = 0$  if  $r > n$ .  $\square$

Prop:  $v \in V \Rightarrow v \wedge : \Lambda^r V \rightarrow \Lambda^{r+1} V$  linear

$\omega \in \Lambda^j V \Rightarrow \omega \wedge : \Lambda^r V \rightarrow \Lambda^{r+j} V$  linear

$(\omega_1 \wedge) \circ (\omega_2 \wedge) = (\omega_1 \wedge \omega_2) \wedge : \Lambda^r V \rightarrow \Lambda^{r+j+k} V$  if  $\omega_1 \in \Lambda^j V$  and  $\omega_2 \in \Lambda^k V$  (associativity)

Remark:  $\Rightarrow \Lambda^* V = \bigoplus \Lambda^r V$  is a ring (Lec. 4).