

From the Past

We have learned to check whether a given function solves a differential equation. Additionally, we now know the solutions to the following differential equations:

- $\frac{dy}{dt} = f(t)$, which we solve by _____ $f(t)$.
- $\frac{dy}{dt} = ky$, which has general solution $y(t) = \underline{\hspace{2cm}}$.
- We know how to check if a given function $y(t)$ is a solution to a differential equation $\frac{dy}{dt} = f(t, y)$ by plugging $y(t)$ into both sides of the equation.

Slope Fields

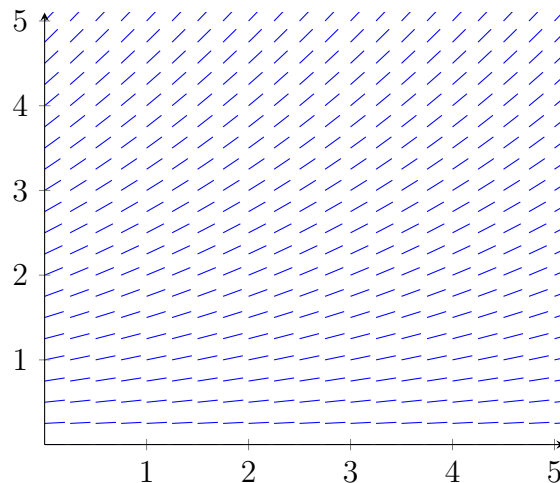
A general (first-order) differential equation can be written as

$$\frac{dy}{dt} = f(t, y),$$

where $f(t, y)$ is some function of t , y or both. This function gives the slope of the function $y(t)$ at the point (t, y) .

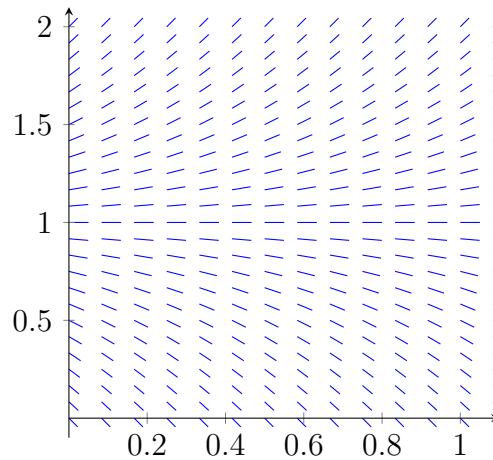
We can get an idea of the general behavior of a differential equation by looking at a plot of the slopes given by $f(t, y)$ at each point. These plots are called *slopefields*.

1. Consider the differential equation $\frac{dy}{dt} = 0.25y$. What is its general solution? (Hint: see the bullet points above.)
2. (a) Slopefields allow us to look at the solution a different way. Below is the slopefield for $\frac{dy}{dx} = 0.25y$, a plot of the slopes, which are represented by short line segments. Sketch the solutions corresponding to $y(0) = 0.5, 1, \text{ and } 3$.

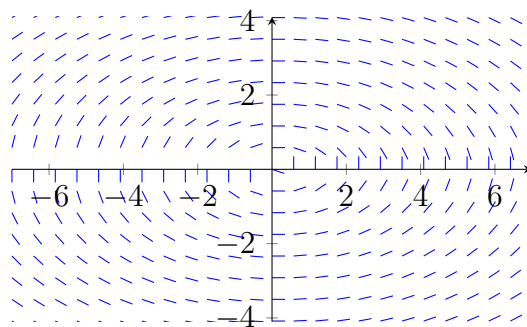


- (b) Notice that if you move horizontally in the slopefield above, all the ticks have the same slope, whereas if you move up and down, the slope changes. Why is this?

3. Let's look now at $\frac{dy}{dt} = 2(y - 1)$ and the initial conditions $y(0) = 0.75$, and $y(0) = 1.25$ on the slopefield below.



- (a) Draw solutions for each of those initial conditions on the slope field.
- (b) What happens to each one of these solutions at $t \rightarrow \infty$? Answer from your drawings!
- (c) What if we have the initial condition $y(0) = 1$? Draw a solution corresponding to such an initial condition. Can you write down a formula for it?
- (d) Note that the slopes in the slopefield only change when we move _____, but not when we move _____. Can you explain this?
4. Below is a slopefield for the differential equation $\frac{dy}{dt} = -\frac{t}{3y}$.



- (a) In the first quadrant the slopes are positive/negative (cross out the wrong answer).

- (b) What about in the other quadrants? Check that this matches up with the differential equation!
- (c) Sketch solutions for each of the following initial conditions: $y(0) = 2$, $y(0) = 4$, $y(3) = 0$, $y(-4) = 1$. Be sure to label which is which.
- (d) What geometric figure do the solution curves look like?
- (e) Are there any constant solutions, such as $y = 1$ was in the previous question? How can you tell just from the differential equation itself?
- (f) Notice that in this case, the slopes are not constant if you move horizontally *or* vertically. Why not? How could you tell this from the differential equation itself?

Equilibrium Solutions

Definition An *equilibrium solution* is a solution (i.e. value of y) that is constant for all values of t .

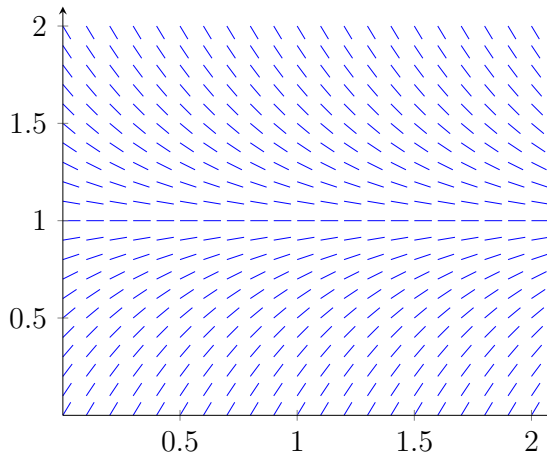
5. Fill in the blanks: The equilibria of a differential equation are the _____ where _____. On the slopefield, they will correspond to a y value at which we see only _____ ticks.

Definition An equilibrium is called *stable* if a small change in the initial condition gives a solution which tends back to the equilibrium at $t \rightarrow \infty$. An equilibrium is *unstable* if a small change in the initial condition gives a solution which veers away from the equilibrium at $t \rightarrow \infty$.

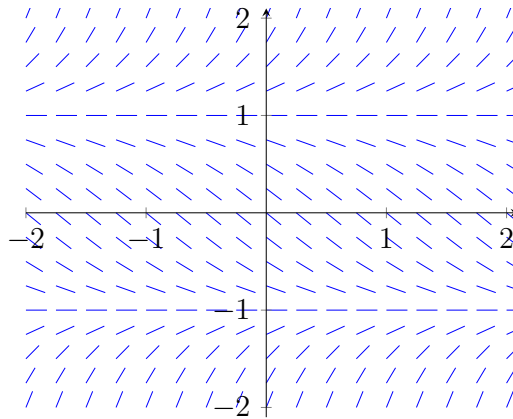
6. (a) Is the equilibrium of the differential equation in Question 3 above stable or unstable? Why?
- (b) Why doesn't the differential equation in Question 4 have any equilibria?

7. For each of the differential equations below, find their equilibrium solutions, then use their slopefields to determine if each equilibrium is stable or unstable.

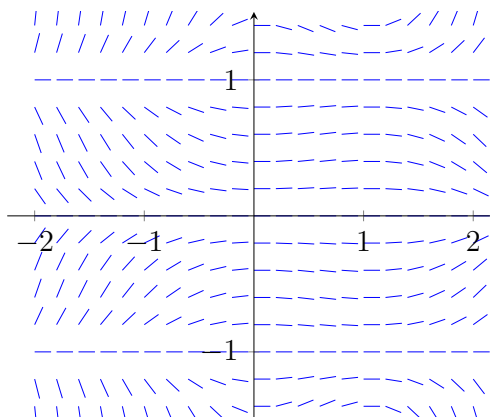
(a) $\frac{dy}{dt} = -2(y - 1)$. (Compare this to Question 3 above!)



(b) $\frac{dy}{dt} = (y - 1)(y + 1)$.

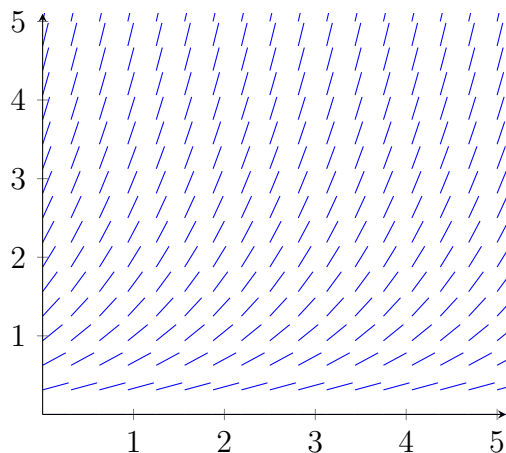


(c) $\frac{dy}{dt} = ty(t - 1)(y - 1)(y + 1)$.

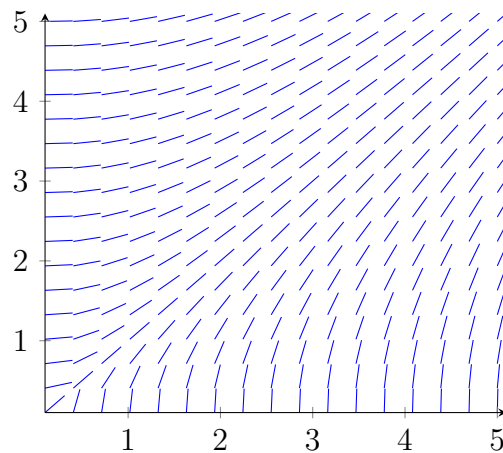


8. For the slope fields below identify the differential equation and sketch three solutions for three different initial conditions.

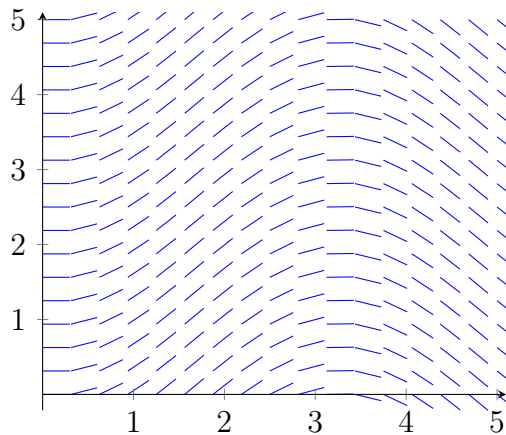
i)



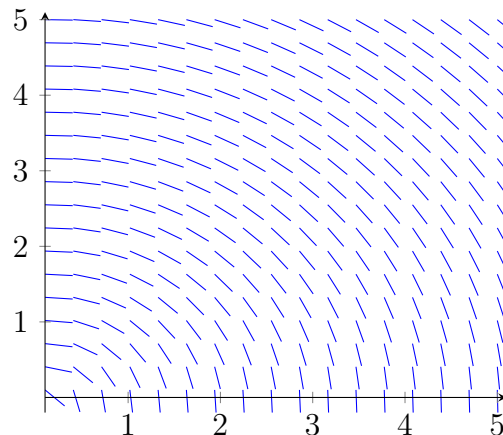
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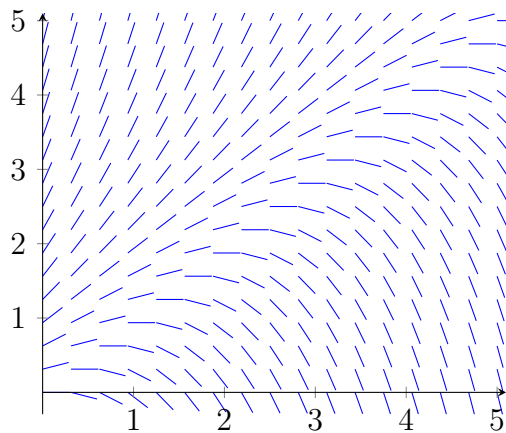
ii)



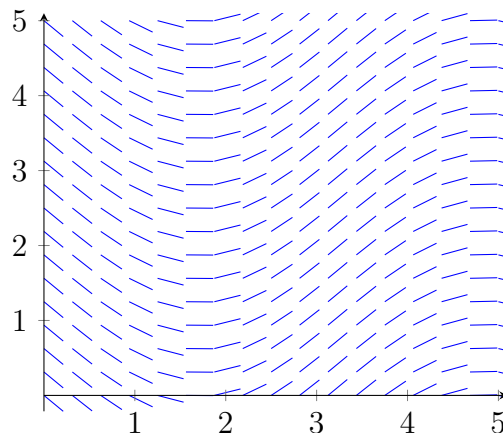
v)



iii)



vi)



- (a) $\frac{dy}{dt} = \frac{t}{y}$ _____ (b) $\frac{dy}{dt} = y$ _____ (c) $\frac{dy}{dt} = \sin(t)$ _____
 (d) $\frac{dy}{dt} = -\frac{t}{y}$ _____ (e) $\frac{dy}{dt} = y - t$ _____ (f) $\frac{dy}{dt} = -\cos(t)$ _____

Important Principle Notice that in all the above slopefields, solutions to the differential equations with different initial values *do not cross each other*. This will be true of all the differential equations we study. Keep this in mind when you do any work with solutions of differential equations!