Previous Differential Equations

We previously found solutions to two classes of differential equations:

- $\frac{dy}{dt} = f(t)$, which we solve by ____________________ $f(t)$.
- $\frac{dy}{dt} = ky$, which has solution $y(t) =$ _______.

Today, we will solve equations of the form

$$\frac{dy}{dt} = \frac{f(t)}{g(y)}.$$

Note that all the above equations are of this form!

The Idea...

Questions

1. Find $\frac{dy}{dt}$ if $\ln y - \frac{1}{2}t^2 = 17$.

2. By reversing the process we used above, let's solve $\frac{dy}{dt} = -\frac{t}{y}$.

Separation of Variables

Let's generalize the process above. Suppose

$$\frac{dy}{dt} = \frac{f(t)}{g(y)}.$$

(In the second problem above, what is $g(y)$?) Also, let $G(y)$ be an antiderivative of $g(y)$.

3. Then,

$$\frac{d}{dy}G(y) = \underline{\hspace{3cm}}$$

and

$$\frac{d}{dt}G(y(t)) = \underline{\hspace{6cm}} = \underline{\hspace{6cm}}$$

Notice, that we can rewrite our differential equation as $g(y)\frac{dy}{dt} = f(t)$. Use the above to rewrite this, with the left hand being a derivative with respect to t.

$$\underline{\hspace{4.5cm}} = f(t)$$

Therefore, $G(y) + C = \displaystyle\int f(t)\,dt$ (Why?). Or, to put it another way,

$$\boxed{\int g(y)\,dy = \int f(t)\,dt}$$

In the differential equation $\frac{dy}{dt} = \frac{f(t)}{g(y)}$, our formula says that we can "cross-multiply," separating out the dy from the dt in the process, then drawing an integral sign. This is a great strength of the notation, but also a great weakness. On the one hand, it's easy to remember, but on the other hand, it's easy to confuse this mnemonic device with actual mathematical content. Don't do so!

Solve the following initial value problems:

4. $\frac{dy}{dt} = \frac{1}{5}(2 - y)$, $y(0) = 3$

5. $\frac{dy}{dt} = \frac{1}{5}(2 - y)$, $y(0) = 1$

6. $\frac{dy}{dt} = y \sin t$, $y(0) = 4$

7. $\frac{dy}{dt} = \frac{y^2}{t}$, $y(1) = 2$

8. Fill in the blanks: We previously saw that $\frac{dy}{dt} = ky$ has solution $y(t) =$ ______. The equilibrium solution is $y =$ ___. This corresponds to $C =$ ___.

   (a) Use separation of variables instead to show that $y = e^C e^{kt}$.

   (b) But if $y = e^C e^{kt}$, then how can $e^C = 0$? Something went wrong here. Why did we lose a solution when we used separation of variables?

Exercises

For each of the following differential equations, find all equilibrium solutions (if any exist) and determine their stability (feel free to use Geogebra to see slopefields). Then solve the differential equation (or initial value problem) by separation of variables.

9. $\frac{dy}{dx} = \frac{x}{y}, y(2) = 4.$

10. $\frac{dy}{dx} = y^2 x$

11. $\frac{dy}{dx} = -(y-1)^3\sqrt{x}$, $y(0) = 4$

12. $\frac{dy}{dx} = x(1+y^2)$

13. $\frac{dy}{dx} = \sqrt{y}x^2, y(0) = 4$ Could $y(0) = -4$? Explain. Careful with this one, especially finding value(s) of C!

14. $\frac{dy}{dx} = \tan(y)x$ (Hint: What is another way to write $\tan(y)$? Also, note that when using the applet, solutions are a mess due to computational issues. Ignore them and just look at the slopes near equilibria to determine stability. You may also want to increase the y range and tick density.)

You can't always solve the equation!

15. Consider the differetial equation $\frac{dy}{dt} = (y+1)(y-1)(y+2)$.

    (a) Show that this equation is seperable, but that actually solving it is not practical.

    (b) What are the equilbria of this equation?

(c) By sketching a slopefield (copy it from the applet – feel free to paste an image in here if you like), determine the stability of each of the equilibria.

(d) On your slopefield above, sketch three non-constant solutions with different behaviors.

16. Do the same for the differential equation $\frac{dy}{dt} = ty(y-2)^2(y+2)$. Note that the slopefield is a bit more complex, since t is part of the derivative.