Introduction

Today, we will model population growth and radioactive decay using a differential equation we know well already:

$$\frac{dy}{dt} = ky.$$

The focus of today is not so much in solving the equation (we know how to do that already,) but rather in understanding why this particular equation models both our situations and in attaching terminology to various features of its solutions.

Review

1. Fill in the blank: the solution of the initial value problem $\frac{dy}{dt} = ky$, $y(0) = y_0$ is $y(t) = $ _____

Population Growth

Suppose that we have a small initial population P_0 on a large piece of land full of resources.

Questions

2. (a) Explain why the following statement makes sense:

   "The rate of growth of the population is proportional to its current size".

   (b) Suppose that the population at time t after the beginning of our observation is given by $P(t)$, and let $P(0) = P_0$. Express the above statement mathematically.

   (c) Solve your initial value problem and explain why this model is called "Exponential Growth".

Let's put some numbers on all this:

3. Suppose that we start off with 20 people and that the population growth rate is $4\% = 0.04$ per year.

   (a) Write down the initial value problem corresponding to this situation and solve it.

   (b) How long does it take for the population to double from 20 people to 40?

   (c) How long does it take to double again from 40 to 80 people?

   (d) If we started off with 100 people instead of 20, how long would it take for the population to double to 200 people?

   (e) Suppose that the population growth rate is $2\% = 0.02$. How long would the population take to double from a people to $2a$ people?

4. As you can see, the amount of time it takes for a given population to double does not depend on the initial population, but only on the _____.

Definition The *doubling time* of an exponentially-growing population is the time it takes for the population to double.

Radioactive Decay

Given a large sample of a radioactive substance, the rate at which the substance decays is proportional to the amount of substance present, measured in number of atoms, or more usefully, in moles.

Suppose that we start with N_0 moles of a radioactive substance, and that the number of moles present at time t is given by $N(t)$.

5. Write down and solve an initial value problem corresponding to the above situation. What is the difference between this initial value problem and the one we considered above for population growth?

6. Suppose that the constant of proportionality for a given radioactive substance is $\lambda = 0.02$ per day. That is, it loses 2% of its amount each day. Calculate the *half-life* of this substance. That is, find the time it takes for the amount of substance to halve.

7. Thorium-234 has a half-life of 24 days.

   (a) If we start with 4 moles of Thorium-234, how long will it take until we only have one mole remaining?

   (b) Find the constant of proportionality λ for the decay model for Thorium-234. By approximately what percentage does the substance decay each day?

8. (a) Suppose that a radioactive substance decays exponentially with constant of proportionality λ. Find its half-life in terms of λ.

   (b) Suppose that a radioactive substance has half-life τ. Find the constant of proportionality for its exponential decay in terms of τ.