

The Exponential Model with Migration

In situations of small populations with near-unlimited resources, we made the assumption that populations grow in proportion to their current size. In other words, that *relative growth rate* is constant:

$$\frac{dP}{dt} = kP.$$

If $P(0) = P_0$, this model has solution

$$P(t) = \underline{\hspace{2cm}}.$$

It is often useful to consider the situation where the population grows exponentially, but where organisms leave at some rate. For example, people may migrate away, or the body may act to kill off bacteria.

1. The population of a certain tropical island is currently 800 and, in the absence of immigration or emigration, increases at a rate proportional to the population itself (with proportionality constant $k = .03$ when time is measured in years). However, each year 36 people emigrate to the mainland. Find and solve an initial-value problem for the population $N(t)$. What is the equilibrium value of the population? Is it stable or unstable? Why does that make sense in terms of population growth and migration? Graph $N(t)$ and find the time when the population becomes zero.

2. Let $B(t)$ denote the number of infectious bacteria in a patient's body. Suppose that, left unchecked, these bacteria would multiply at a rate proportional to the number there (with constant of proportionality equal to 1.6 if time is measured in hours). Suppose that the patient's immune system kills off 800 of these bacteria per hour. Suppose that at $t = 0$ the patient has B_0 bacteria in his system. For which values of B_0 will the immune system eventually conquer the infection? Analyze the stability of the equilibrium in terms of the infection.

3. In a certain polluted lake, the fish population is decreasing at a rate proportional to its size (with constant of proportionality equal to .04 when t is measured in months). However, environmentalists add 2,000 fish per month to restock the lake. Considering the stability of the equilibrium, if the environmentalists continue this restocking for a long time, how many fish will be in the lake? Is this dependent on the initial population?

The Logistic Model with Migration

When resources are limited, we used the *logistic model* of population growth, which assumes that instead of staying constant, the relative growth rate of the population decreases as population increase. From this, we derived the logistic equation:

$$\frac{dP}{dt} = kP \left(1 - \frac{P}{L} \right).$$

If $P(0) = P_0$, this has solution

$$P(t) = \frac{L}{1 + Ae^{-kt}} \quad \text{with} \quad A = \frac{L - P_0}{P_0}.$$

Suppose that a population grows logistically, but that a certain proportion of the population migrates away per unit time. Note that in contrast to the situation above, we look at a *proportion* of the population migrating away, rather than an absolute number. The absolute number migration situation is dealt with in lab this week.

Questions The population of wolves in a forest grows according to a logistic model:

$$\frac{dP}{dt} = 0.25P \left(1 - \frac{P}{2500} \right)$$

Suppose that a wildlife bridge is built over a nearby highway, resulting in 10% of the wolf population migrating away each year.

4. (a) The current population is P . 10% of the current population leaving each year means we should subtract $0.1P$ from the logistic model above. Write a new differential equation.

- (b) Simplify your equation to show that we still get a logistic model. What is the new carrying capacity?

5. Suppose that we are in the situation described in Question 4. Suppose also that the initial population of wolves is 1,000 (before the bridge is opened). The wildlife bridge is opened when there are 2,000 wolves in the forest. Draw a graph of the wolf population vs. time. On your graph, both the initial carrying capacity and the new one. Also mark inflection points (if any) and the point when the bridge is opened.