A Summary of the Story So Far

Motivated by finding distance traveled from velocity, we've been trying to compute areas under curves. So far, we've been able to estimate these areas by approximating parts of the area with rectangles, computing their areas, and adding them all up.

The Definite Integral - A First Definition

$\displaystyle\int_a^b f(x)\,dx$ (read this: the integral from a to b of $f(x)$ dee x) is the exact area between the curve $f(x)$ and the x-axis from $x = a$ to $x = b$, where area below the axis counts as negative.

1. By drawing the appropriate graphs and using some geometry, compute the following definite integrals:

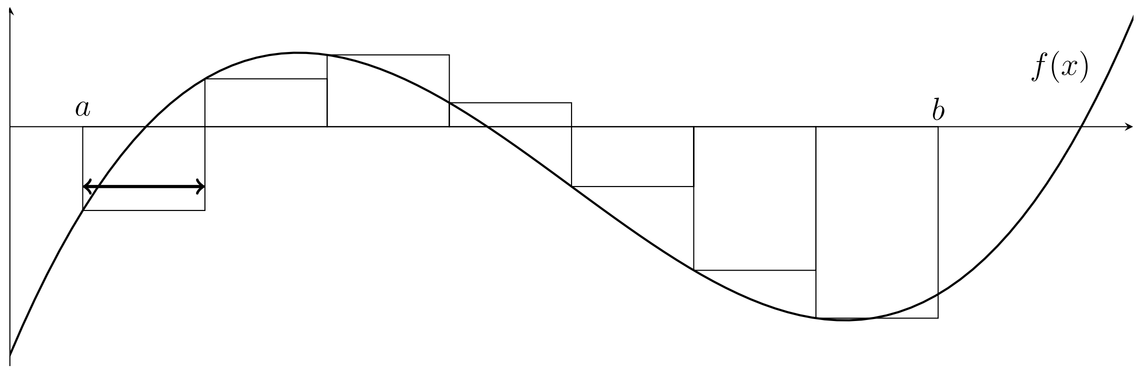
   (a) $\displaystyle\int_{-1}^{3} 6\,dx$

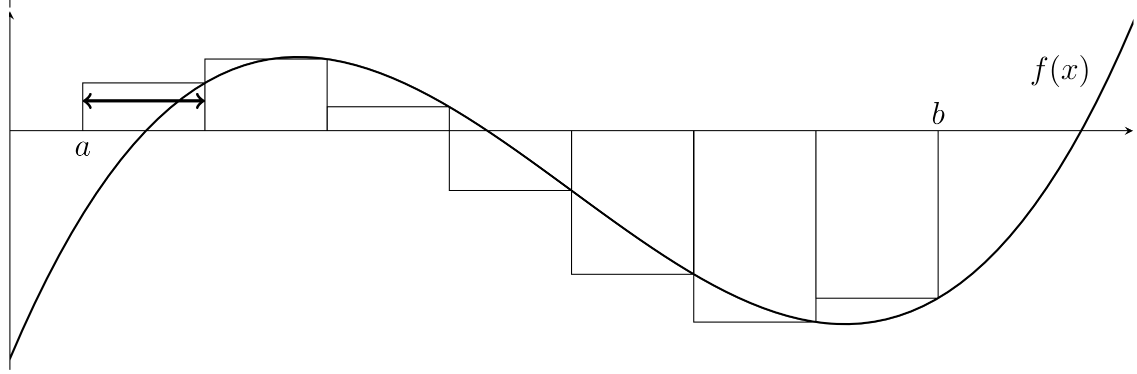
   (b) $\displaystyle\int_{-1}^{6} 2x\,dx$

Our aim is to compute any more complicated integrals, such as $\displaystyle\int_1^4 \frac{1}{x}\,dx$, the area under $f(x) = \frac{1}{x}$ between $x = 1$ and $x = 4$. Let's start with estimating, and seeing where that takes us...

Riemann Sums

Consider the following graphs of a function $f(x)$, and the area underneath $f(x)$ between $x = a$ and $x = b$, which is divided into n rectangles of equal length Δx:





2. (a) One of the graphs represents a left-hand sum and one is a right-hand sum. Label them.
   (b) Recall that we divide the interval $[a, b]$ into n subintervals *of equal length*, Δx. In terms of a, b, and n, what is Δx? Label the appropriate distance on each of the graphs Δx.

3. (a) In terms of a and Δx, write down the *fourth* subinterval: [__________, __________].
   (b) In terms of a, i, and Δx, write down the i^{th} subinterval: [__________, __________].

4. (a) In a LHS, what is the height of the rectangle drawn on top of the fourth subinterval? Write your answer in terms of f, a, and Δx.

   Height of rectangle on fourth subinterval in LHS: _____________.

   (b) In a LHS, what is the height of the rectangle drawn on top of the i^{th} subinterval? Write your answer in terms of f, a, and Δx.

   Height of rectangle on i^{th} subinterval in LHS: _____________.

   (c) In a LHS, what is the area of the rectangle drawn on top of the fourth subinterval? Write your answer in terms of f, a, and Δx.

   Area of rectangle on fourth subinterval in LHS: _____________.

(d) In a LHS, what is the area of the rectangle drawn on top of the i^{th} subinterval? Write your answer in terms of f, a, and Δx.

Area of rectangle on i^{th} subinterval in LHS: ____________.

5. (a) In a RHS, what is the height of the rectangle drawn on top of the fourth subinterval? Write your answer in terms of f, a, and Δx.

   Height of rectangle on fourth subinterval in RHS: ____________.

   (b) In a RHS, what is the height of the rectangle drawn on top of the i^{th} subinterval? Write your answer in terms of f, a, and Δx.

   Height of rectangle on i^{th} subinterval in RHS: ____________.

   (c) In a RHS, what is the area of the rectangle drawn on top of the fourth subinterval? Write your answer in terms of f, a, and Δx.

   Area of rectangle on fourth subinterval in RHS: ____________.

   (d) In a RHS, what is the area of the rectangle drawn on top of the i^{th} subinterval? Write your answer in terms of f, a, and Δx.

   Area of rectangle on i^{th} subinterval in RHS: ____________.

6. Label the fourth subinterval and the corresponding rectangle with all the information from parts (a) and (c) of questions 4 and 5 on the appropriate graphs above.

Remember that our approximations of the area under the curve were merely the sum of the areas of the rectangles in each case. It's about time we write down a compact expression for these sums:

7. Using your answers in 4(d) and 5(d), write down sums in Σ-notation for the LHS and RHS, each on n subintervals. Your answers should contain n, f, Δx, and some index (say i):

   LHS = __________________ RHS = __________________

The Exact Area Under a Curve - The Expression

8. (a) What happens to our two estimates of the area under a curve as we increase n?

   (b) How big do we want to take n to get the exact area under the curve? (Hint: this is a bit of a trick question...)

9. Using the RHS from the last question on the previous page, write down an expression giving the exact area under a curve $f(x)$ between $x = a$ and $x = b$.

The Definite Integral - A Proper Definition

Suppose $f(x)$ is function on an interval $[a, b]$. Then the *definite integral* of $f(x)$ from $x = a$ to $x = b$ is the area under the curve of $f(x)$ from $x = a$ to $x = b$:

$$\int_a^b f(x)\, dx = \underline{\hspace{6cm}},$$

where $\Delta x = \underline{\hspace{2cm}}$.

When you've understood this definition, pat yourself on the back – we've been working toward it for the entire week. It is the central object of study of Calculus II!

Of course, we still have no frickin' clue how to compute these objects. Therefore: to be continued....