Review: Properties of Integrals

- $\int_a^a f(x)\,dx = 0$
- $\int_a^b f(x)\,dx = -\int_b^a f(x)\,dx$
- $\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx$
- $\int_a^b (f(x) \pm g(x))\,dx = \int_a^b f(x)\,dx \pm \int_a^b g(x)\,dx$
- $\int_a^b cf(x)\,dx = c\int_a^b f(x)\,dx$
- If $f(x) \le g(x)$ for $a \le x \le b$, then $\int_a^b f(x)\,dx \le \int_a^b g(x)\,dx$
- If $m \le f(x) \le M$ for $a \le x \le b$, then $m(b-a) \le \int_a^b f(x)\,dx \le M(b-a)$

1. Suppose $F(x)$ is an antiderivative for $f(x)$. What does this mean?

Indefinite Integrals

Let's introduce some new notation. Let $\int f(x)\,dx$ represent all possible antiderivatives for $f(x)$. We call this the **indefinite** integral.

2. (a) If $F(x)$ is an antiderivative for $f(x)$, why is $F(x)+C$ also an antiderivative, where C is any constant?

   (b) How many possible antiderivatives does a function have and how are they connected?

3. Give as many differences between $\int_a^b f(x)\,dx$ and $\int f(x)\,dx$ as you possibly can.

4. Fill in the following basic integration formulas. Here, k and n are constants.

- $\int \sin x\,dx =$ ____________ (Hint: what function's derivative is $\sin x$?)
- $\int \cos x\,dx =$ ____________
- $\int k\,dx =$ ____________
- $\int e^{kx}\,dx =$ ____________
- $\int x^n\,dx =$ ____________ for $n \neq -1$

5. (a) For the last integral above, why do have to stipulate that $n \neq -1$?

   (b) If $x > 0$, what is $\int \frac{1}{x}\,dx$? Why must we stipulate $x > 0$ here?

   (c) If $x < 0$, show that $\ln(-x)$ is an antiderivative for $\frac{1}{x}$. Why is $\ln(-x)$ defined in this case?

   (d) Explain why this shows that $\int \frac{1}{x}\,dx = \ln|x| + C$.

We also have the properties

$$\int (f(x) + g(x))\,dx = \int f(x)\,dx + \int g(x)\,dx$$

and

$$\int cf(x)\,dx = c\int f(x)\,dx.$$

It's worth your time to think about why these are true.

More Questions...

6. Evaluate $\int \sin\left(-\frac{x}{2}\right) + e^{17x}\,dx$ and check your answer.

7. (a) To find $\int e^{2x}\ dx$, you might first guess that an antiderivative is $e^{2x} + C$. But $\frac{d}{dx}e^{2x} = 2e^{2x}$. How could you use this to find the correct antiderivative?

   (b) Does a similar procedure work for $\int e^{-x^2}\ dx$? A first guess might be e^{-x^2}. But $\frac{d}{dx}e^{-x^2} = -2xe^{-x^2}$. So is $\frac{e^{-x^2}}{-2x}$ an antiderivative of e^{-x^2}? What's the difference from the last question?

8. (a) What is $\int \frac{1}{1+x^2}\,dx$? (Hint: for this and the next two, think back to the beginning of the semester!)

   (b) What is $\int \frac{1}{\sqrt{1-x^2}}\,dx$?

   (c) What are two rather different (maybe!) antiderivatives of $-\frac{1}{\sqrt{1-x^2}}$? What must be true about the relation between the two? Draw graphs to show this.

Summary

For now, we can only find antiderivatives of functions that we can recognize as the derivative of something else. For example, we know that the derivative of e^{kx} is ke^{kx}, so we can conclude that the function $\frac{1}{k}e^{kx}$ has derivative e^{kx}. Therefore, $\frac{1}{k}e^{kx}$ is an antiderivative of e^{kx}:

$$\begin{aligned} \frac{d}{dx}e^{kx} &= ke^{kx} \\ \Rightarrow \frac{1}{k}\frac{d}{dx}e^{kx} &= e^{kx} \\ \Rightarrow \frac{d}{dx}\frac{1}{k}e^{kx} &= e^{kx} \\ \Rightarrow \int e^{kx}\,dx &= \frac{1}{k}e^{kx} + C. \end{aligned}$$

However, if we can't easily find a function whose derivative is what we're given, we don't know how to find an antiderivative. Next week, we'll figure out how to reverse the product rule and chain rule to add to our library of antiderivatives.

Having said that, there are many functions for which it is very hard (or even impossible– in the case of e^{-x^2}, for example) to write down 'nice' antiderivatives. What exactly 'nice' means, and how we can get around that, at least for theoretical applications, will be covered at the end of Week 9.