Introduction

1. Let $f(x) = 2x$. Consider the following function:

$$F(x) = \int_0^x 2t\, dt.$$

   (a) Draw a graph of $f(x)$. On it, illustrate $F(1)$, $F(2)$ and $F(3)$.

   (b) What does $F(x)$ measure?

   (c) Without further computation, how do you know $F(x)$ is increasing? Concave up?

   (d) Calculate $F(1)$, $F(2)$, $F(3)$ and $F(-1)$.

   (e) What function do you think $F(x)$ is?

   (f) Use the FTC to prove your hypothesis from the previous question, then fill in the blanks below:

$$\frac{d}{dx}\int_0^x 2t\, dt = \underline{\qquad\quad}, \text{ so } \int_0^x 2t\, dt \text{ is an } \underline{\qquad\qquad\qquad} \text{of } f(x) = 2x.$$

   (g) Is $G(x) = \displaystyle\int_2^x 2t\, dx$ also an antiderivative of $f(x) = 2x$? If so, what constant do $F(x)$ and $G(x)$ differ by?

Stuff from the Past that will be Important Today!

2. $\star$ **Extreme Value Theorem (105L Worksheet 11-3):** If $f(t)$ is continuous on the closed interval $[a, b]$, then it has a __________ __________ and a __________ __________ on that interval.

   $\star\star$ **Bounding Integrals (106L Worksheet 6-3):** If $m \leq f(t) \leq M$ for $a \leq t \leq b$, then

$$\leq \int_a^b f(t)dt \leq$$

FTC II

> **The Second Fundamental Theorem of Calculus**
> Let f be continuous on an interval. Then for x and a in that interval
>
> $$\frac{d}{dx}\int_a^x f(t)dt = f(x)$$

3. **Proof:** Suppose $f(t)$ is a continuous function and let $g(x) = \displaystyle\int_a^x f(t)dt$.

   (a) Then

$$\frac{g(x+h)-g(x)}{h} = \frac{\qquad\qquad}{h} = \frac{\qquad\qquad}{h}$$

   (b) By ($\star$), since $f(t)$ is a continuous on $[x, x+h]$, then it attains a smallest value m and a largest value M. Then, by ($\star\star$),

$$\leq \int_x^{x+h} f(t)dt \leq$$

   (c) Dividing everything by h, we get

$$\underline{\quad m \quad} \leq \frac{\int_x^{x+h} f(t)dt}{h} \leq \underline{\quad M \quad}$$

   (d) As $h \to 0$, what happens to the interval $[x, x+h]$?

   (e) As $h \to 0$, what happens to m and M?

   (f) Therefore, $\displaystyle\frac{d}{dx}g(x) = \lim_{h\to 0}\frac{\int_x^{x+h} f(t)\,dt}{h} =$ __________.

4. Recall that we were never able to antidifferentiate e^{-x^2}. Can you now write down an antiderivative for it?

5. Find the following derivative in two different ways: $\dfrac{d}{dx}\displaystyle\int_2^x \cos t\, dt$

   (a) Using FTC I:

   (b) Using FTC II:

6. Let $g(x) = \displaystyle\int_1^x \sqrt{1+t^2}\, dt$.

   (a) What is $g'(x)$?

   (b) What is $g(x^3)$?

   (c) What is $\frac{d}{dx}g(x^3)$? (Hint: you need the chain rule here. This is a composite function.)

7. (a) Find a function $g(x)$ such that $g'(x) = \sqrt{1+x^2}$ and $g(2) = 0$.

   (b) Find a function $g(x)$ such that $g'(x) = \sqrt{1+x^2}$ and $g(2) = 10$.

8. Let $g(x) = \int_0^x f(t)\,dt$, with $f(x)$ is continuous. Crost out the wrong answer for each of the following:

   - If $f(x) > 0$ and $x > 0$, then $g(x)$ is positive/negative and increasing/decreasing.
   - If $f(x) > 0$ and $x < 0$, then $g(x)$ is positive/negative and increasing/decreasing.
   - If $f(x) < 0$ and $x > 0$, then $g(x)$ is positive/negative and increasing/decreasing.
   - If $f(x) < 0$ and $x < 0$, then $g(x)$ is positive/negative and increasing/decreasing.

9. What constant do the following two antiderivatives of $f(x)$ differ by?

$$F(x) = \int_{-1}^{x} f(t)\ dt \qquad G(x) = \int_{1}^{x} f(t)\ dt$$

   (Hint: draw pictures, and see Worksheet on Properties of Integrals, property 4!)