Modeling with Trigonometric Functions

Pre-Lab: Review of Transformations

Graphing and Transforming Trigonometric Functions

1. Match the following transformations with the description of their effects. (Assume $k > 1$. Refer to the 'New Functions from Old' worksheet from 105L if needed!)

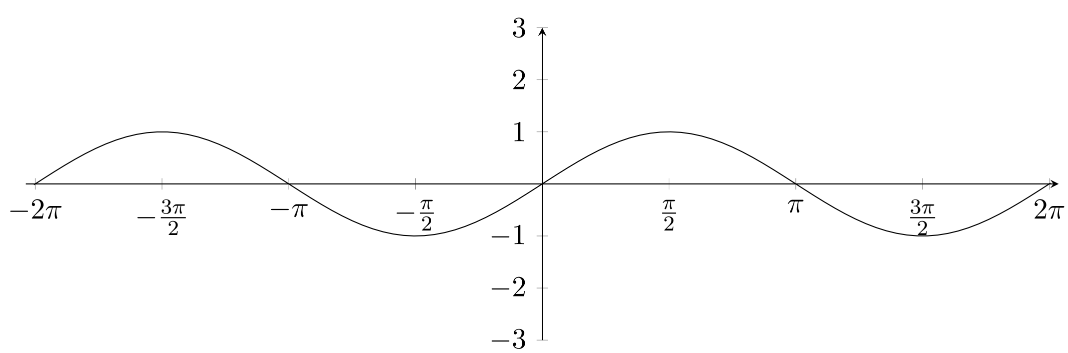
(a) $f(x-k)$ (b) $f(x)+k$ (c) $f\left(\frac{1}{k}x\right)$ (d) $kf(x)$

(e) $f(x+k)$ (f) $f(x)-k$ (g) $f(kx)$ (h) $\frac{1}{k}f(x)$

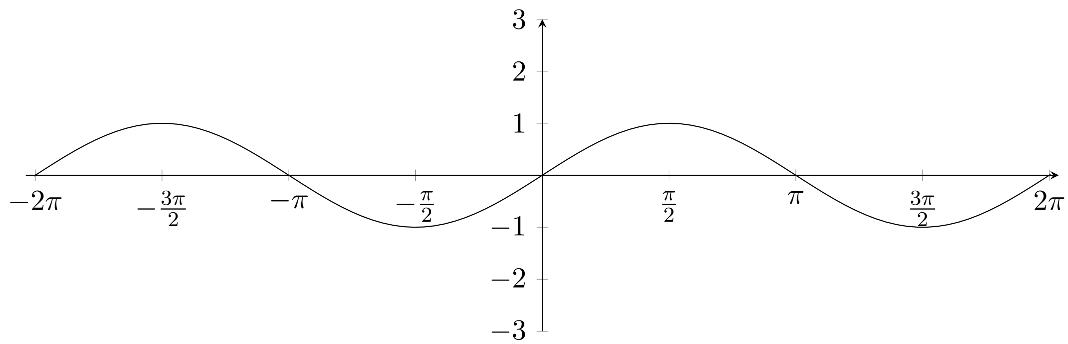
Vertical shift k units down	Horizontal stretch by a factor of k
______	______
Vertical shrink by a factor of k	Horizontal shift k units to the left
______	______
Vertical shift k units up	Horizontal shrink by a factor of k
______	______
Vertical stretch by a factor of k	Horizontal shift k units to the right
______	______

Before we begin constructing our model, it is important to understand some basic features of the sine function. The graph of $\sin x$ is drawn repeatedly below. For each graph, add in the given transformations of $\sin x$ in different colors. Label them carefully. In each case, describe the difference between $\sin x$ and your graphs.

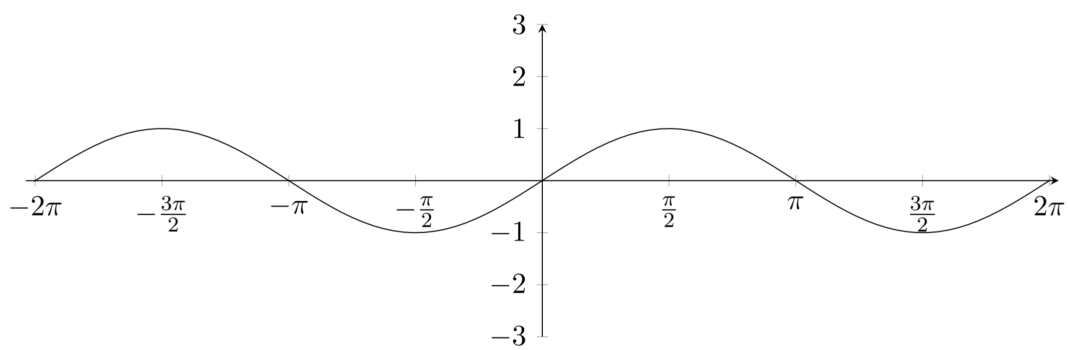
2. Graph $\sin\left(x+\frac{\pi}{2}\right)$ and $\sin(x-1)$.



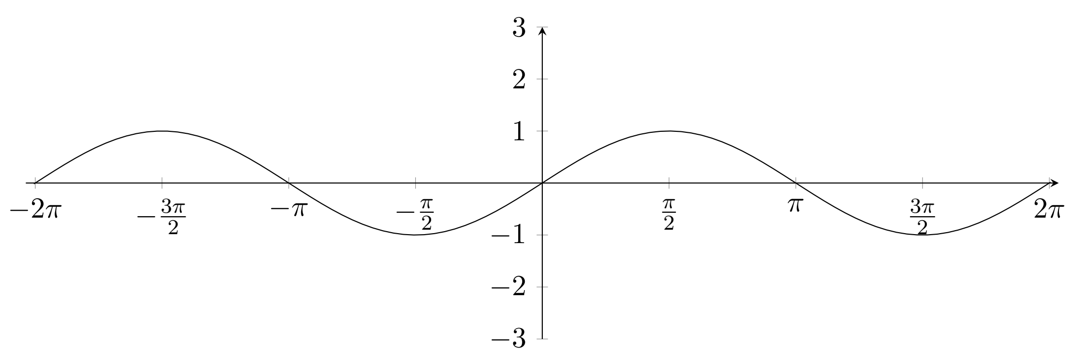
3. Graph $3\sin x$, $0.5\sin x$, $-\pi\sin x$.



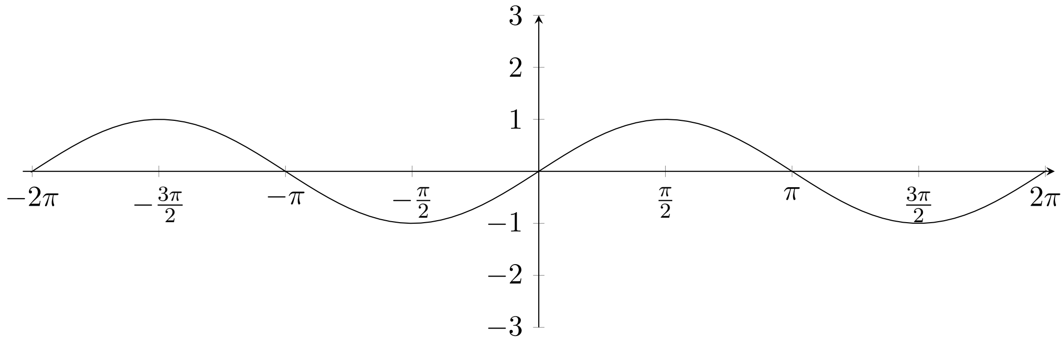
4. Graph $\sin(3x)$, $\sin(0.5x)$, $\sin(-\pi x)$.



5. Graph $\sin(0.5x) - 2$.

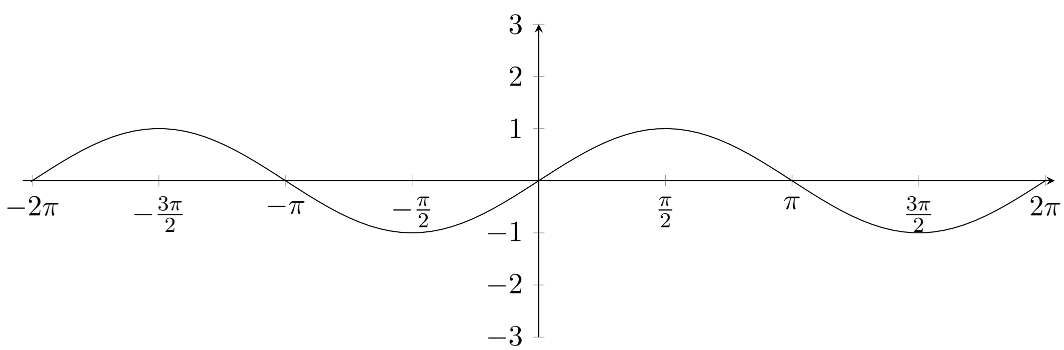


6. Graph $\sin(2x-2)$ and $\sin(2(x-2))$. What is the period of each of your two new graphs? How far right are each of them shifted?



Period is π for both. First is shift right by 1, second by 2.

7. Graph $-2\sin\left(x+\frac{\pi}{2}\right)+1$.



Part I: Daylight Hours for Lisbon

The following table gives the length of the day (that is, the number of hours of daylight) for Lisbon, Portugal.

Date	Day	Hours of Daylight
1/2	2	9.1
1/18	18	9.5
2/19	50	10.7
3/7	66	11.4
3/23	82	12.2
4/24	114	13.7
5/10	130	14.3
5/26	146	14.8
6/11	162	15.1
6/27	178	15.1
7/13	194	14.9
7/29	210	14.4
8/14	226	13.8
9/15	258	12.4
10/1	274	11.7
10/17	290	10.9
11/18	322	9.5
12/4	338	9.2
12/20	354	9.0

This data is reproduced in the spreadsheet for this lab.

1. Open the spreadsheet for this lab, make a copy, and rename it with your group names.
2. Plot the above data as a scatter plot.

The shape of your scatter plot should look approximately like a sinusoidal function. Let's try to model the daylight data by a function of the form

$$f(x) = A\sin[B(x - C)] + D.$$

Our aim is to determine the values of A, B, C, and D.

3. **Calculation of** A The number A is called the *amplitude* and it is equal to half the difference between the highest point and the lowest point on the curve. Since we don't have data for every day of the year, we don't really know on which days the highest and lowest points occur and what their exact values are. By looking at the data and its plot, explain what might be reasonable values for the highest and lowest points. Then calculate A in cell $G2$.

4. **Calculation of B** The value of B affects the period of the function (how often the function repeats its standard pattern). If we increase B, the function shrinks. If we decrease B, the function stretches. The period of $\sin x$ is 2π.

   What is the period of the daylight function? Calculate the value of B in cell $G3$. Explain your formula.

5. **Calculation of D** The number D determines the position of the daylight function along the y-axis with respect to the position of the function $\sin x$. Decide which vertical direction and by how many units we need to shift $\sin x$ in order to match the daylight function. Calculate the value of D in cell $G5$. Explain your formula.

6. **Calculation of C** The number C is called the phase shift and it determines the position of the daylight function along the x-axis with respect to the position of the function $A\sin(Bx)+D$. Decide which horizontal direction and how many units we need to shift $\sin x$ in order to match the daylight function. Calculate the value of C in cell $G4$.

7. Add data for your completed model in Column D, and insert a new chart plotting both the data and the model as lines[1]. Make sure you use entries in column G so you can easily change your model if it doesn't match.

8. There are actually other values of A, and C that will produce the same model function $f(x)$ (even though the constants differ, the function $f(x)$ will be the same at every value of x). Write down two of these different forms of $f(x)$, one with the same value for A that you have above, and one with the negative of your value of A above. You can test out your answers by changing A and C in their cells!

[1] Unfortunately, Google Sheets does not have the option of making a mixed scatter and line plot. Excel does, so if you know Excel well, feel free to use it. Best you can do in Google Sheets is add point markers to one of the lines.

Part II: Daylight Hours for Stockholm

The table below gives the length of the day (that is, the number of hours of daylight) for Stockholm, Sweden. The data is reproduced in the second tab of your spreadsheet.

Date	Day	Hours of Daylight
1/2	2	4.6
1/18	18	5.8
2/19	50	9.2
3/7	66	10.9
3/23	82	12.6
4/24	114	16.1
5/10	130	17.9
5/26	146	19.6
6/11	162	20.8
6/27	178	21.0
7/13	194	20.0
7/29	210	18.4
8/14	226	16.6
9/15	258	13.2
10/1	274	11.5
10/17	290	9.8
11/18	322	6.4
12/4	338	5.0
12/20	354	4.3

Carry out the same analysis as you did in Part I for this data, using the second tab of your spreadsheet.

Part III: Tides in the Bay of Fundy

On March 18, 2018, the high tide in the Bay of Fundy was at midnight and the water level then was 26 feet. The very next tide was 6 hours and 48 minutes later and the water level then was 2 feet. Assuming the water level $y(t)$ varies sinusoidally, answer the following questions:

1. Find a formula involving a sin function for $y(t)$.

2. Find a formula involving a cos function for $y(t)$.

Note that we only need a successive maximum and minimum value for $y(t)$ to create our sinusoidal model here.

Report: Global Warming

In the third tab of the spreadsheet for the lab, you will find 75 years of daily maximum temperature data for Raleigh (from NOAA, the National Oceanic and Atmospheric Adminstration[2]). I ran a algorithm that searches for the best function of a given form to model the data[3], getting:

$$T(t) = 19.25\cos\left(\frac{2\pi}{365.25}(t - 197.5)\right) + 71 \text{ degrees F.}$$

1. In column E, compute the values of $T(t)$ over the 75 years given. Pick a period of five years from the middle of the data and plot both the data and the model's output for that date range. You should see that while the data is noisy, overall, the model captures the trend well.

2. Explain what each of the four parameters means in this context: What does $2 \times 19.25 = 38.5$ represent? Why is the period not 365 days? What does that 197.5 mean? Explain how you would estimate D from the the data. Do so in the spreadsheet to verify your understanding.

3. In Column D, you will find the average temperature for each date (for example, cell D2 contains the average temperature on Jan 1 over many years). Add this data to your plot in a different color from the model. You should notice that they are very close to each other.

The model above assumes that the average daily maximum temperature is constant over the period of time studied. If global warming exists, we would expect the average maximum daily temperature to *increase* over time instead. So instead of D being constant, perhaps it can be modeled with a linear function:

$$A\cos(B(x - C)) + Mx + D_{lin}.$$

Running the algorithm on this, I got a new model:

$$T_{\text{lin}}(t) = 19.25\cos\left(\frac{2\pi}{365.25}(t - 197.5)\right) + 0.00006533t + 70 \text{ degrees F.}$$

4. Explain why most of the parameters did not change.

5. In Column F, compute the value of $T_{\text{lin}}(t)$ over the 75 years. Plot first five years and last five years for the two functions and compare them. What do you notice? Explain your observations.

6. The slope of the linear part of T_{lin} is 0.00006533. What are the units of this slope? What does the fact that it is positive mean? Use it to compute the average increase of daily maximum temperature per decade.

NOAA states that globally, temperature has increased $0.13°F$ per decade since 1880, and $0.32°F$ per decade since 1981[4]. If you did your work correctly, your estimate should be pretty squarely in the middle of that!

Each group should hand in a document with the plots (well labeled, titled, etc.), as well as all answers the questions above. All answers should be in full sentences and your report should be self-contained. That is, a reader should be able to understand it without reference to the lab.

[2] https://www.ncdc.noaa.gov/cdo-web/datasets

[3] This is *non-linear regression*, often called *non-linear optimization*.

[4] https://www.climate.gov/news-features/understanding-climate/climate-change-global-temperature