Riemann Sums

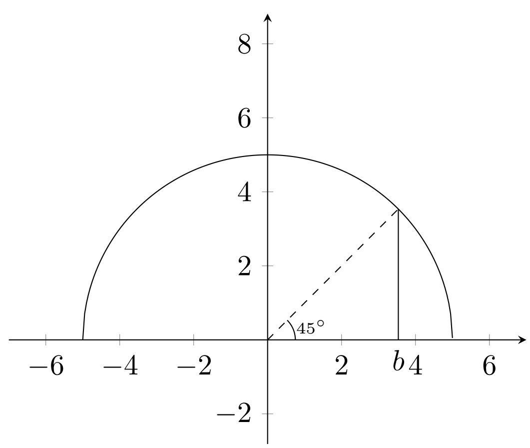
Purpose:

The purpose of this lab is to review the ideas, discussed in class, concerning how to compute sums of the form $\sum_{i=1}^{n} f(a + i\Delta x)\Delta x$ and to explore what happens as we compute those sums for large and larger n.

Open the spreadsheet for this lab. Then go to the File menu, make a copy, and rename the sheet with your group names.

Part I - Area of Part of a Circle

1. Consider the area above the x-axis and under the graph of $y = \sqrt{25 - x^2}$ between $x = 0$ and $x = b$ as shown in the picture below. Note that this is $\int_0^b \sqrt{25 - x^2}\, dx$ (we'll compute the value of b in a second).



   (a) Carefully copy the above picture and shade in the area under discussion. Make sure to check this with your TA before proceeding.

   (b) What is the value of a? Store it in cell B1.

   (c) Find the value of b and store it in cell B2.

(d) Find the exact area of the region using what you know about geometry and store it in cell B7 (be sure to enter the formula you used to compute it, so it's as precise as possible).

(e) Let's take $n = 5$ rectangles, enter 5 in cell B3, and enter a formula into cell B4 that computes Δx.

(f) Draw a picture showing the Right Hand Sum (RHS) for $n = 5$.

2. Recall that the i^{th} interval in a Riemann sum is [______________, ______________]. Use the following steps to compute left-hand and right-hand sums for this integral with $n = 5$:

   - In Column F, enter i values from 0 to 5.
   - In cell G2, enter a formula that computes $a + i\Delta x$ for the given i in column F. Recall that you previously stored a and Δx in cells B1 and B4 respectively. So that things don't go wrong later, be sure to refer to B1 as B$1 and to B4 as B$4.
   - Copy your formula down from cell G2 to fill in cells G3 through G7.
   - In cell H2, compute the height of the first rectangle in a left hand sum (that is, $f(a)$). Note that you have a in cell G2!
   - Copy your formula down to cells H3 through H7.

3. We will now find the areas of rectangles in the LHS and the RHS, then add them up. It turns out we don't have to duplicate most of the work!

   - In cell I2, compute the area of the first rectangle in a LHS. Then, in cell I3, compute the area of the second rectangle in a LHS. How does this relate to the area of the the first rectangle in a RHS?
   - In the rest of Column I, find the areas of rectangles by multiplying your height from column H by Δx. Once again, refer to Δx as B$4, not just B4.
   - We want to compute the LHS in cell L1. Which five areas should we add to do this? Enter a formula in L1 that does this.
   - We want to compute the RHS in cell L2. Which five areas should we add to do this? Enter a formula in L2 that does this.
   - In cell L6, enter a formula for the average of the LHS and RHS. In Cell L7, enter a formula for the difference of this average from the true value.

4. Next, we will compute the *errors*. That is, how far off from the true area our sums are.

   - Since our function is decreasing, should the LHS overestimate the true value of the integral or underestimate it? What about the RHS?
   - In cell L3, enter a formula for the difference between the left hand sum in cell L1 and the true value (from cell B7). Do the same for the right hand sum in cell L4. Check that the first is positive and the second negative. Why must this be the case?
   - Compute the error for the average in cell L7. It should be negative. We will come back to this later in the lab.

5. Record your answers for LHS and RHS, as well as their respective errors, in the first row of the table in rows 13-19 in the spreadsheet. Do the same for the average of the LHS and RHS and its error. To make sure we have as accurate an answer as possible, use Ctrl-C (or Command-C or a Mac) to copy L2 (the LHS), then use Ctrl-Shift-V (or Command-Shift-V) to paste it in. This copies the value rather than the formula for it.

6. Next, modify your spreadsheet to complete the rest of the table. For each row, you'll need to:
   - change the value of n (which will automatically change Δx);
   - add more values of i;
   - copy down heights and areas;
   - and modify your formulas for summing up for the LHS and RHS.

7. By looking at the LHS and RHS columns of your table, estimate $\lim_{n\to\infty}\sum_{i=1}^{n} f(a+i\Delta x)\Delta x$ correct to one decimal place. Enter your estimate in cell L9. What do you think the exact value is?

Part II - Some More Areas

In this part, we will repeat the above computations for two more functions.

8. Suppose $f(x) = e^{-x}$. We will estimate the area under the curve between $x = 1$ and $x = 2$.
   (a) Graph $f(x) = e^{-x}$ between $x = 0$ and $x = 3$. Draw a picture that shows the Left-Hand Sum for the area under $f(x)$ between $x = 1$ and $x = 2$ with $n = 5$ subintervals.
   (b) Use the second tab of the spreadsheet to compute left-hand and right-hand sums for this integral. Complete the tables in that tab.
   (c) If $f(x) = e^{-x}$, estimate $\lim_{n\to\infty}\sum_{i=1}^{n} f(a+i\Delta x)\Delta x$ correct to three decimal places.

9. Suppose $f(x) = \sin(x)$. We will estimate the area between the curve and the x-axis from $x = \frac{\pi}{2}$ to $x = 2\pi$.
   (a) Graph $y = \sin(x)$ between $x = 0$ and $x = 2\pi$ and consider the area between the x-axis and the graph between $x = \frac{\pi}{2}$ and $x = 2\pi$. Draw a picture that shows the Left-Hand Sum for $n = 6$.
   (b) Do you expect our estimates for this area to be positive or negative? Explain.
   (c) Use the third tab of the spreadsheet to compute Riemann sums for this integral. Complete the tables in that tab.
   (d) What do you think $\lim_{n\to\infty}\sum_{i=1}^{n} f(a+i\Delta x)\Delta x$ is?
   (e) Looking at your graph and your answer to the previous question, can you figure out what the area under the graph of $\sin x$ between $x = 0$ and $x = \frac{\pi}{2}$ is? Explain.

Part III - Midpoint and Trapezoid Sums

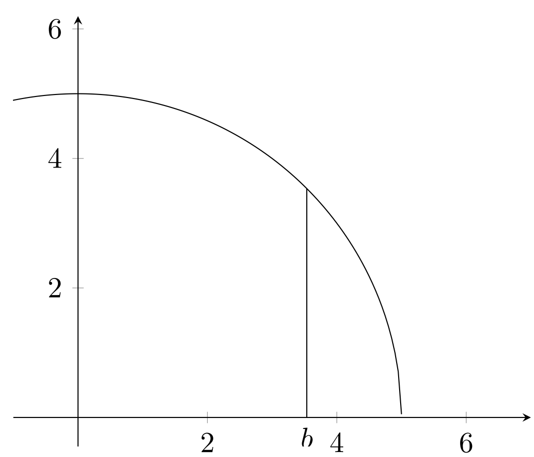
We can also compute an estimate of the area using the following sum:

$$\sum_{i=1}^{n} f\left(a+\left(i-\frac{1}{2}\right)\Delta x\right)\Delta x.$$

This is the *Midpoint Sum* or MPS. Instead of using the left or the right endpoint of the interval, this estimate uses the midpoint of the interval.

10. Explain why, if $1 \leq i \leq n$, then $a+(i-\frac{1}{2})\Delta x$ gives the midpoint of the i^{th} interval. (Hint: multiply out those parentheses, then think about how many steps of length Δx from a your formula shows you.)

11. On a copy of the following axes, draw a picture of this sum, for $n = 5$.



12. Modify your spreadsheet to compute the mid-point sum with five intervals. Note that you will need to compute the new x values (use the formula for the midpoints of the intevals above), then the height of the rectangles, then their areas. Also find the error from the exact answer. Fill your answers in the last two columns of the table that begins on row 23. Then do the same for the larger values of n. Again, be sure to use copy by value to record your answers accurately.

13. The average of the LHS and RHS for a given n is called the *Trapezoid Sum*. You computed trapezoid sums for this integral in Part I. Compare the averages of the LHS and RHS to the MPSs. Are they the same? Answer in terms of over and underestimates.

Part IV: The Definite Integral

Definition: The *definite integral* of f from a to b is

$$\int_a^b f(x)dx = \lim_{n\to\infty} \sum_{i=1}^{n} f(a + i\Delta x)\Delta x,$$

where $\Delta x = \frac{b-a}{n}$. If f is continuous then this limit always exists.

Note also that $\int_a^b f(t)dt = \lim_{n\to\infty} \sum_{i=0}^{n-1} f(a + i\Delta x)\Delta x = \lim_{n\to\infty} \sum_{i=1}^{n} f\left(a + \left(i - \frac{1}{2}\right)\Delta x\right)\Delta x$. That is, you can estimate integrals using LHS, RHS, MPS, or Trap. For continuous functions, as $n \to \infty$, all of them converge to the same value.

14. Using only the work done in parts I-III of this lab, find (precisely if possible, otherwise to three decimal places)

    (a) $\int_0^b \sqrt{25 - x^2}dx$, where $b = \frac{5}{\sqrt{2}}$. (b) $\int_1^2 e^{-x}dx$. (c) $\int_0^\pi \sin x dx$.

15. Note that this definition allows for $f(x) < 0$ for some x in the interval $[a, b]$. Use some geometry and logic (i.e. no Fundamental Theorem) to find:

    (a) $\int_0^{2\pi} \sin x dx$. (b) $\int_1^4 2 - x dx$.

Part V: Overestimates and Underestimates

16. Let's examine whether, given certain conditions, our various sums overestimate or underestimate $\int_a^b f(x)dx$. Fill in the table with the words 'overestimate' and 'underestimate'. Explain your answers.

	$f'(x) > 0, f''(x) > 0$	$f'(x) < 0, f''(x) > 0$	$f'(x) > 0, f''(x) < 0$	$f'(x) < 0, f''(x) < 0$
RHS				
LHS				
Trap				
MPS				

Report

In this report, we will analyze and attempt to understand how the error in each of our estimates changes as we increase n. Do questions 1-3 using the first integral from the lab:

$$\int_0^{\frac{5}{\sqrt{2}}} \sqrt{25 - x^2}dx.$$

1. **Preliminaries**
   (a) What is the exact value of this integral? Explain your answer completely. (See Question 1 from the lab.)
   (b) Which of the LHS, RHS, MPS, and Trap estimates of the integral do you expect to be overestimates? Which are underestimates? In terms of properties of the curve, explain. (See Part V of the lab.)

2. **Comparing Errors**

   We defined the *error* of a particular estimate to be

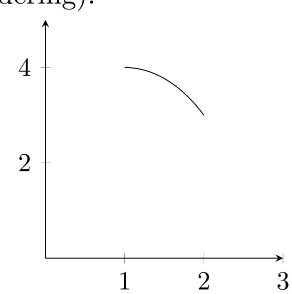
   $$error = estimate - true\ value.$$

   That is, if we have an overestimate, our error is positive.

   (a) Copy your table of estimates and errors from the first tab of the spreadsheet into a new tab.
   (b) In new columns, compute each of the following:
   - The ratio of the RHS errors to the LHS errors for every value of n;
   - The ratio of the TRAP errors to the MPS errors for every values of n.

   (c) Fill in the blanks in the following sentences:
   - The ratio of RHS to LHS errors is approximately _______. That is, the errors of LHS and RHS are approximately ______________ in magnitude.
   - The ratio of TRAP to MPS errors is approximately _______. That is, the error for TRAP is approximately ______________ as large as the error for MPS.

   (d) i. Draw two copies of the curve shown below (which is decreasing and concave down, like the function we are considering).

   ii. On the interval from $x = 1$ to $x = 2$, draw an LHS and an RHS rectangle on one copy of the graph, and an MPS and a TRAP trapezoid on the other. (See worksheet 5-3 for help with the latter).

iii. Shade in the errors for the LHS and RHS on your first copy, and the errors for MPS and TRAP on the other. Hopefully, you can see that the errors for LHS and RHS are approximately equal, and that the error for TRAP is approximately twice the error for MPS!

3. **Error and the number of subintervals**

   (a) In a new column, compute the *successive ratios* of errors for each of your four estimation methods. That is, compute $\frac{\text{LHS}_{10}}{\text{LHS}_5}$, $\frac{\text{LHS}_{20}}{\text{LHS}_{10}}$, and so on. Repeat for RHS, MPS, and TRAP.

   (b) Fill in the following blanks:

   - When we double n (the number of subintervals), the errors for LHS and RHS are reduced by a factor of approximately ______;
   - When we double n (the number of subintervals), the errors for TRAP and MPS are reduced by a factor of approximately ______.

   (c) Suppose we increase the number of subintervals from $n = 10$ to $n = 50$. By appoximately what factor will the errors for LHS and RHS decrease? What about the errors for TRAP and MPS?

4. **Integrating from data**

   When we gather experimental data, we often do not have a formula for a function that fits it. If we want to compute the area under the curve of the data, we need to estimate the definite integral.

   Consider the following data:

t	0	1	2	3	4	5
$f(t)$	2.3	5.2	7.6	3.1	4.2	6.0

   (a) Compute LHS_5, RHS_5, and TRAP_5 for $\displaystyle\int_0^5 f(t)\,dt$.

   (b) Why is it not possible to decide which of these are overestimates and which are underestimates?

   (c) We saw above that midpoint sums are the most accurate estimate of the value of a definite integral. Why is it impossible to compute MPS_5 for the integral given just this data?

   (d) Fill in the blanks: Suppose the experiment was redone so that values of $f(t)$ were measured every half second over the interval from zero to five seconds. This would allow us to compute LHS, RHS, and TRAP with $n =$ ______, and MPS with $n =$ ______. So, while it is true that MPS_n has half the error as TRAP_n for a given value of n, this shows that the two are not really comparable. Instead, we should compare MPS with n intervals to TRAP with ______ intervals.

For the report, hand in all tables from the questions above, as well as answers to all questions. Be sure to write in full sentences and explain your work carefully. For the tables, show all LHS and RHS data to three decimal places, and TRAP and MPS to five decimal places.