

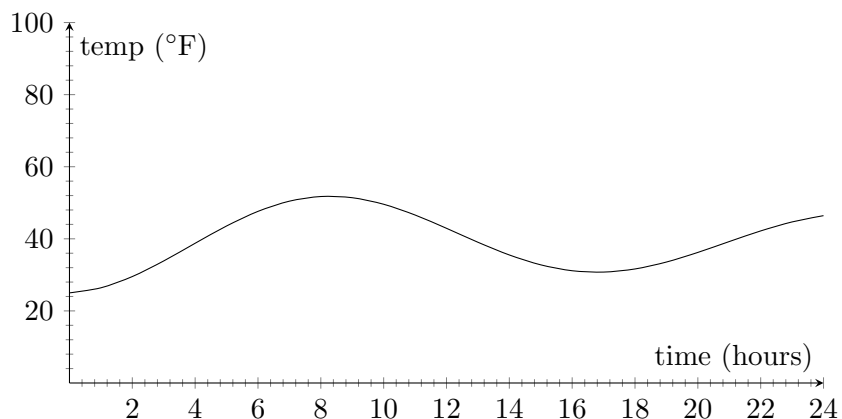
Average Value of a Function

Purpose

The purpose of this lab is to understand how the definite integral can be used to find the average value of a function $f(x)$ over an interval $[a, b]$.

Part I: Average Value of a Function

Suppose that we have a device that will allow us to make a continuous record of the outside temperature over a 24 hour period. On a typical winter day in Durham, North Carolina, our device might produce a graph like the one pictured below.



1. By studying the temperature graph, what would you estimate as the average temperature for the 24 hour period shown? Draw a dotted horizontal line through the temperature graph at your estimated average temperature. Looking at the graph do you think you made a reasonable estimate? Explain.

One could also estimate the average temperature by using the graph to find the temperature at regularly spaced times and then averaging these temperatures.

2. Use the graph to estimate the outside temperature at two hour intervals. Find the average of these temperatures. How does this estimate compare to your estimate made in question 1?

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If we let divide the 24 hour period into n equally spaced times, each of length $\Delta t = \frac{24}{n}$:

$$0 + \Delta t, 0 + 2\Delta t, 0 + 3\Delta t, \dots, 0 + n\Delta t$$

and we call our temperature function T , then

$$\begin{aligned} \text{average temperature} &\approx \frac{T(0 + \Delta t) + T(0 + 2\Delta t) + T(0 + 3\Delta t) + \dots + T(0 + n\Delta t)}{n} \\ &= \frac{1}{n} \sum_{i=1}^n T(0 + i\Delta t) \end{aligned} \tag{1}$$

- How could one improve this estimate for the average temperature?
- Since $\Delta t = \frac{24}{n}$, solving for n in terms of Δt yields $n = \underline{\hspace{2cm}}$.
- Substitute your expression for n into the estimate for average temperature in equation 1 and show

$$\text{average temperature} \approx \frac{1}{24} \sum_{i=1}^n T(0 + i\Delta t) \Delta t.$$

- Why does it make sense to take n toward ∞ now?
- As $n \rightarrow \infty$ (or $\Delta t \rightarrow 0$) we have

$$\text{average temperature} = \lim_{n \rightarrow \infty} \frac{1}{24} \sum_{i=1}^n T(0 + i\Delta t) \Delta t. \tag{2}$$

- Write the above expression (2) for the average temperature as a definite integral.
- What is the relationship between the divisor 24 and the definite integral?

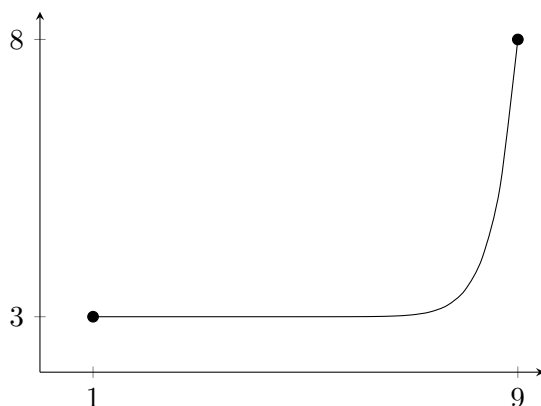
You have established that to find the average temperature one needs to integrate the temperature function over the given interval and then divide the result by the length of the interval. This expression for average temperature can be extended to finding the average value of any continuous function over an interval. The average value of the function f over the interval $[a, b]$ is given by

$$\text{Average value of } f \text{ on } [a, b] = \frac{1}{b-a} \int_a^b f(t) dt.$$

Part II: Applying the Average Value Formula

The following problems make use of the formula for average value. You may use the fundamental theorem or an area formula from geometry to evaluate the integrals.

8. Find the average value of the function $y = 2x - 6$ over the interval $[3, 5]$. Also compute its average over $[0, 3]$, then over $[0, 5]$.
9. Find the average value of the function $y = \sin(x)$ over the interval $[0, 2\pi]$. Explain intuitively why the answer you gave makes sense.
10. (a) Sketch the graph of $y = \sqrt{1 - x^2}$ over the interval $[0, 1]$.
 (b) Would you expect the average of $y = \sqrt{1 - x^2}$ over the interval $[0, 1]$ to be greater than $1/2$, less than $1/2$, or equal to $1/2$. Why? (Hint: add a line from $(0, 1)$ to $(1, 0)$ to your graph. What is its average value?)
 (c) Find the average value of the function $y = \sqrt{1 - x^2}$ over the interval $[0, 1]$.
11. Consider the function pictured below. Explain why the average value of this function over $[1, 9]$ is not 5.5. Is the average greater than or less than 5.5? Explain.



12. Suppose you are driving down a highway at a constant velocity of 50 mph. After 2 hours your car experiences engine trouble and you must reduce your speed to 30 mph. After another hour the engine trouble worsens and you are forced to drive at 20 mph for 30 minutes after which you pull over for repairs.
 - (a) Sketch a graph of your velocity versus time for the 3.5 hour period described above.
 - (b) Explain why your average velocity during the period described above is not
$$\frac{50 \text{ mph} + 30 \text{ mph} + 20 \text{ mph}}{3}.$$
 - (c) Calculate your average velocity for the period described above.

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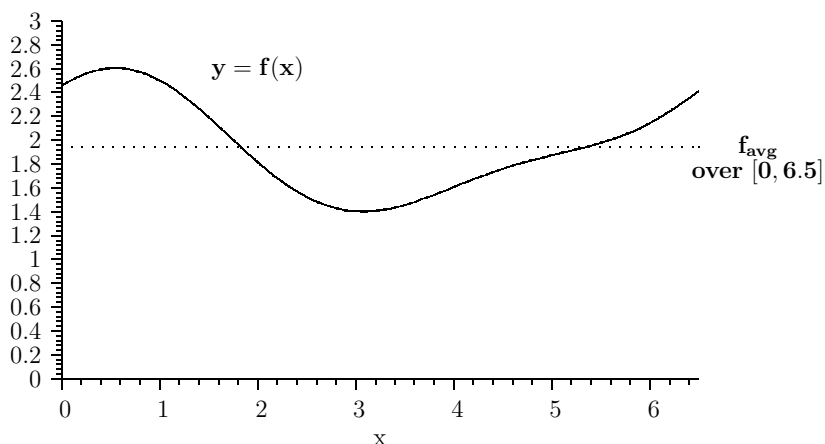
13. The formula for finding the average value of a function f over an interval $[a, b]$ is given by

$$f_{\text{avg}} = \frac{1}{b-a} \int_a^b f(t) dt.$$

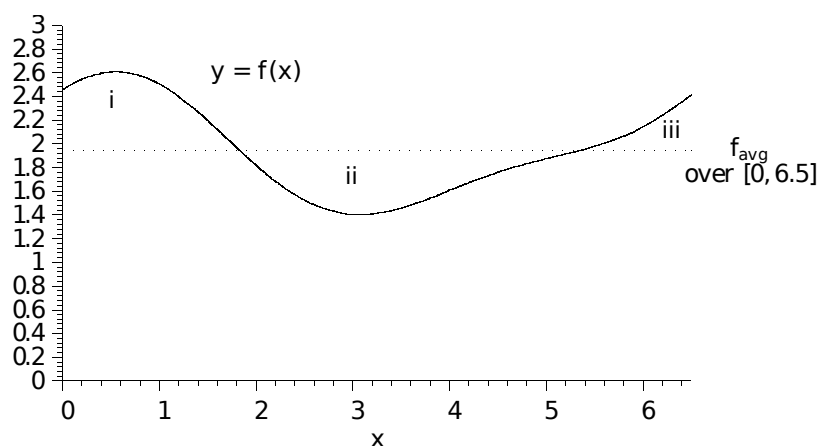
Multiplying both sides of this equation by $(b-a)$ yields

$$f_{\text{avg}}(b-a) = \int_a^b f(t) dt. \quad (3)$$

Pictured below is the graph of f over $[0, 6.5]$. The average value of f is indicated on the graph with a dotted horizontal line. Explain the meaning of equation (3) in terms of the areas of certain regions on the graph. Shade the areas which are equal as expressed by equation (3).



14. Consider the areas between the graph of f and the average value line pictured below. Indicate which regions have equal area. Explain



Report

Hand in complete answers to Questions 5-8 and 12-14. Show all work and write in full sentences.