Quantities of Varying Density

Purpose: The purpose of this lab is to show how a certain quantity whose density varies, such as population, can be calculated using integration.

Introduction: The strategy we will employ for finding a population whose density varies follows the five steps listed below.

1. Divide the region where the population exists into small pieces so that on each piece the population density is approximately constant on each piece.
2. Estimate the population on each piece assuming the density is constant on that piece.
3. Approximate the total population by adding up the contributions to the population from each piece.
4. Turn the sum from step 3 into a definite integral by taking a limit as the number of subdivisions of the region increases to infinity and their size shrinks to 0.
5. Evaluate the definite integral.

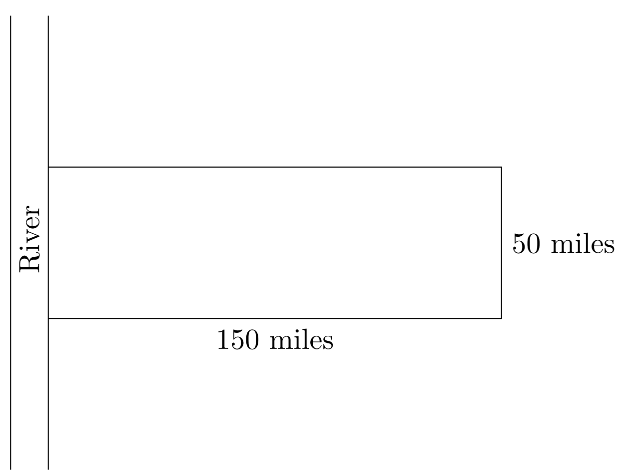
Examples of the units used for population density are people per square mile, mosquitoes per square mile, or plankton per cubic meter.

Part I: Mosquitoes in the Park

Pictured below are the borders of a wildlife park . You can see that it is in the shape of a rectangle with a river flowing along one of its boundaries. By inspecting mosquito traps set up around the park, an entomologist has formulated the following as a model of mosquito concentration.

$$m(x) = \frac{8,000,000}{3x + 120}$$

where x is the distance from the river measured in miles and $m(x)$ is the mosquito concentration measured in mosquitoes per square mile.



1. Open the spreadsheet for this lab, make a copy, and rename it with your group names.

2. Where in the park are the mosquitoes most concentrated? Least concentrated?

3. What is the range of the mosquito concentration in the park? Include units in your answer. Fill in cells B9 and B10 in the spreadsheet, and insert formulas in cells C9 and C10 to calculate this data.

$$\_\_\_\_\_\_\_\_\_\_ \leq m(x) \leq \_\_\_\_\_\_\_\_\_\_$$

4. Divide the park into 5 rectangles of equal width, parallel to the river.
   - Enter 5 for n, the number of rectangles, in cell I3.
   - Enter rectangle numbers from 1 to 5 in column A of your spreadsheet.
   - In column B, compute the distance from the river of the midline of each rectangle. Your formula should involve the rectangle numbers from column A.
   - Using the mid-line of each rectangle as its approximate distance from the river, enter a formula for the approximate density of mosquitoes on that rectangle in column C.
   - All the rectangles have the same height. Enter this height for all the entries in column D.
   - In Column E, calculate the width of each rectangle. Your formula should involve n from cell I3.
   - In Column F, calculate the area of each rectangle.
   - In column G, calculate the approximate number of mosquitoes on each rectangle.
   - Enter a formula to find the approximate total number of mosquitoes in the park in cell I4.

5. Why would it have been a mistake to divide the park into rectangles running perpendicular to the river?

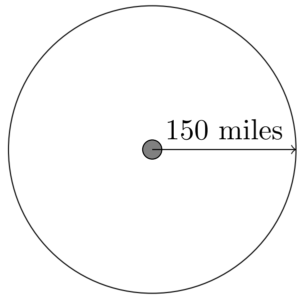
6. How could we improve the estimation of the park's mosquito population made in question 4?

7. Suppose you divide the park into n rectangles of equal width ($\Delta x = 150/n$). Then the distance from the river of the midpoint of the i^{th} rectangle is __________. Write down a Riemann sum (in Σ notation) which yields an estimate of the total number of mosquitoes in the park based on the n rectangles. To get an estimate of total number of mosquitoes on a rectangle, be sure to multiply the approximate density on that rectangle (in mos/sq mil) by its area (in sq mil). In your summation, label these two parts.

8. Set up and evaluate an integral which will yield the total number of mosquitoes in the park. Use the Fundamental Theorem of Calculus to evaluate your integral. Show your work (Answer: $\approx 207,752,616$ mosquitoes)

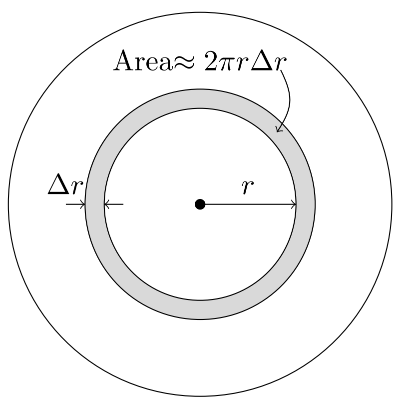
Part II: Circular Park

For this problem we will assume that the wildlife park is in the shape of a circle with radius 150 miles. There is a small pond at its center. We will use the same density function $m(r) = \dfrac{8000000}{3r + 120}$ as we did in Part I, but this time r represents distance from the center of the park.



9. How should we divide the area of the circular park so that the population density of mosquitoes is approximately constant on each piece? For example, slicing the park into wedge shaped pieces like a pizza would be a bad idea. (Why?) Draw a picture showing how you would divide the park.

10. Explain why the area of the shaded piece of the park pictured below can be approximated by $2\pi r \Delta r$ if we assume that Δr is "small." Why is $2\pi r \Delta r$ only the approximate area?



11. Divide the circular park into 5 concentric rings of equal width. Use the second tab of the spreadsheet to do a similar analysis to the one you did in question 4. Note that while your densities should be the same, the areas of each section should now vary!

12. Write down a Riemann sum, with n subdivisions, which approximates the mosquito population of the circular park. Carefully label the radius of each segment, the area of each segment, and the density of mosquitoes on each segment.

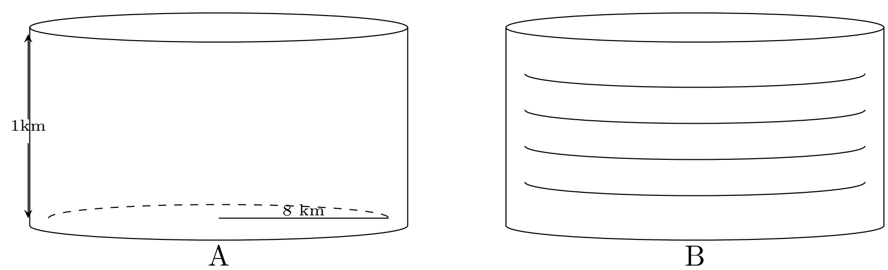
13. Write down a definite integral which yields the mosquito population of the circular park. Evaluate your integral using substitution. (Answer: 1,468,995,577 mosquitoes)

Part III: Pollution Over a City

In this problem you will calculate the amount of pollution in the air over a city. Suppose that environmental engineers have analyzed the air over a city at various heights above the ground and have recorded the results of their study in the table below. The data is reproduced in the third tab of your spreadsheet. Pay careful attention to the units given in the table and on the diagram.

height above the ground in meters	0	200	400	600	800	1000
density of pollutant in kg/m^3	.00040	.00024	.00015	.000089	.000054	.000033

Picture the air above this city as a circular cylinder with radius 8 km and height 1 km as shown below in figure A.



14. Since the density of the pollutant varies with altitude we begin by dividing the cylinder of air horizontally into 5 equal slices as shown above in figure B.

    (a) Find the volume (m^3) of a single slice of air depicted in figure B. Store your result in cell B9 of the third tab. Recall that the volume of a cylinder is $V = \pi r^2 h$, and be careful with units!

    (b) Use the data provided to find a lower and upper estimate for the amount of pollutant (kg) in the cylinder of air above the city in cells B11 and B12 respectively.
    (Answer: upper estimate $\approx 3.75 \times 10^7$ kg)

15. The exponential function $p(h) = 0.0004e^{-0.0025h}$ provides an excellent model for the data given in the table, where h is height above the ground in meters and $p(h)$ is the density of the pollutant in kg/m^3 at altitude h.

    (a) Using the function $p(h)$ write a Riemann sum which approximates the amount of pollutant in the cylindrical column of air, assuming we have divided the cylinder into n slices of equal thickness. Label the volume of each segment, and the density of pollutant on it.

    (b) Write down an integral which yields the total amount of pollutant in the air over the city to an altitude of 1,000 meters. Use the Fundamental Theorem of Calculus to evaluate the integral. (Answer: $\approx$ 29,529,241 kg)

    (c) Pollution exists above 1,000 meters, of course. We could calculate it up to 2,000 meters, or 3,000, but each of these would be arbitrary cutoffs. Instead, perhaps the total amount of pollutant over the city reaches a limit as the height of the cylinder of air increases? By taking the upper bound of your integral to be a variable, u, compute the total amount of pollution up to height u. Does the pollution reach a finite limit?